

MATHEMATICS

ANALYSIS AND APPROACHES - SL

Bill Blyth, Györgyi Bruder,
Fabio Cirrito, Millicent Henry,
Benedict Hung, William Larson,
Rory McAuliffe, James Sanders.
6th Edition

FOR USE WITH THE I.B. DIPLOMA PROGRAMME

MATHEMATICS SL

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TEXT BOOK



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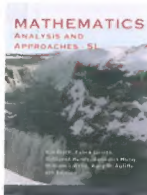
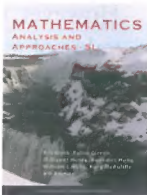
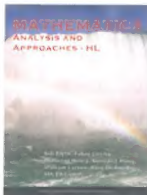
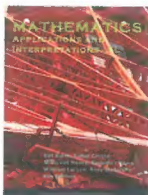
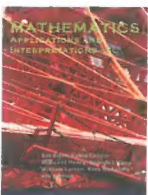
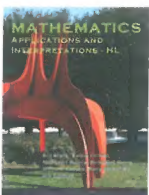
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Using the 6th Editions of IBID

Press Mathematics Texts

This series of texts has been written for the IB Courses Mathematics: Analysis and Approaches and Mathematics: Applications and Interpretations that start teaching in August 2019.

Course Studied	Common Core	Mathematics: Analysis and Approaches (SL)	Mathematics: Analysis and Approaches (HL)	Mathematics: Applications and Interpretations (SL)	Mathematics: Applications and Interpretations (HL)	Discounted Package
Mathematics: Analysis and Approaches (SL)						Pure(SL)
Mathematics: Analysis and Approaches (HL)						Pure(HL)
Mathematics: Applications and Interpretations (SL)						Applied(SL)
Mathematics: Applications and Interpretations (HL)						Applied(HL)

PREFACE

This text for the Mathematics: Analysis and Approaches course has been prepared to closely align with the current course.

It has concise explanations, clear diagrams and calculator references.

Appropriate, graded exercises are provided throughout.

Also relating to International Perspectives and the Theory of Knowledge, it provides more than just the basics. It is an essential resource for those teachers and students who are looking for a reliable guide for their SL course.

This is a re-worked and revised edition of the Standard Level text first published by IBID Press in 1997.

2nd Edition published in 1999

3rd Edition published in 2004

4th Edition published 2012

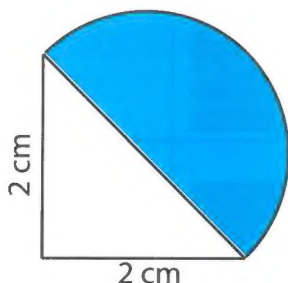
5th Edition published in 2018

6th Edition published in 2019

Rounding

When carrying an answer from one part of a calculation to a subsequent part, it is best to use unrounded values. For example:

A semicircle is constructed on the hypotenuse of a 2 cm right angled triangle. Find its area correct to 4 significant figures.



Stage 1:

Calculate the length of the hypotenuse using the theorem of Pythagoras:

$$\sqrt{2^2 + 2^2} \approx 2.828(4 \text{ s.f.})$$

Stage 2

Find the area of the circle and halve it:

$$\text{Area} = \frac{1}{2} \pi r^2 \approx \frac{1}{2} \pi \times 2.828^2 \approx 12.56 \text{ cm}^2 (4 \text{ s.f.})$$

In this text, we will show calculations in this way as, we believe, students will be able to follow our explanations more easily if we do this.

However, if using a calculator, the best procedure is to use the memory to store a full accuracy version of the radius (second line).

	Line	Des	Norm1	d/c	Real
$\sqrt{(2^2 + 2^2)}$					
Ans→R				2.828427125	
$.5 \times \pi \times R^2$				2.828427125	
				12.56637061	
▶MAT/VCT					

Rounding this gives the answer 12.57 cm^2 (4 s.f.)

Note that the two answers are different. We understand that both answers are usually marked correct in examinations. However, we suggest that using the memory and the calculator value of π (not 3.14) is the better method.

Calculators

Students who are thoroughly familiar with the capabilities of their model of calculator place themselves at a considerable advantage over students who are not.

In preparing a text such as this, we cannot provide an exhaustive account of every place in which a calculator can help. Or, for that matter, an explanation of how each model works!

This text uses examples from Texas Instruments and Casio graphic calculators.

The manufacturers all provide extensive 'manuals'. These can be intimidating.

We suggest that a good strategy is to take each topic and, as you are learning it, take some time to discover your model's capability in that topic.

For example, Section 1.3 deals with counting principles. It is highly likely your calculator will be very helpful here. A good strategy can be to 'Google' or 'Bing' your model plus the topic.

There are now a number of training videos available on YouTube.

Answers

Answers to the Exercises are available as a free download from the publisher's website:

www.ibid.com.au

Also, there are QR codes embedded in the text that link directly to these.

Online Errata



Supplementary Material



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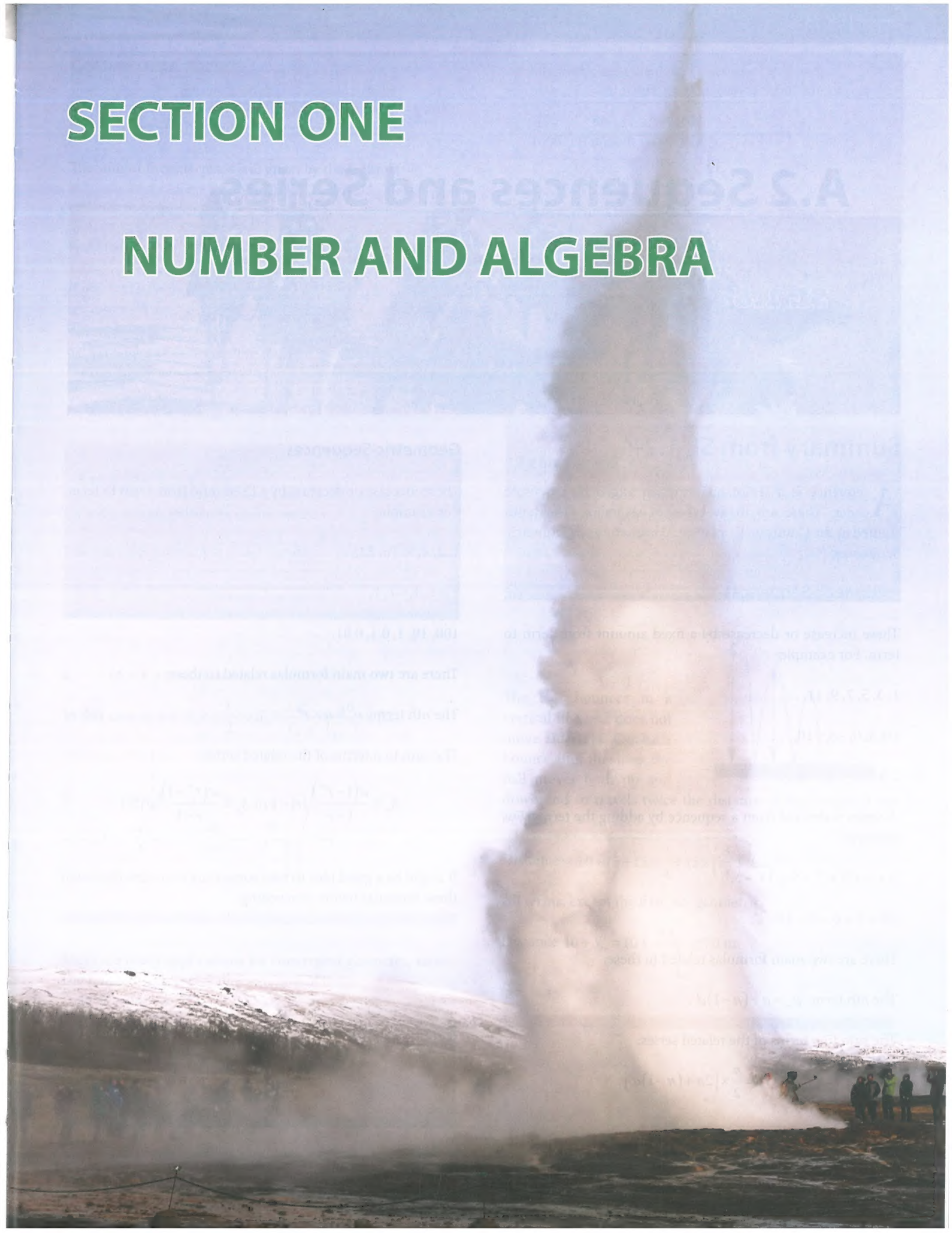
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SECTION ONE

NUMBER AND ALGEBRA



A.2 Sequences and Series

SL 1.8

Summary from SL 1.2-4

A sequence is a set of quantities arranged in a definite order. There are many types of sequence. The types studied in the Common Core were Arithmetic and Geometric Sequences.

Arithmetic Sequences

These increase or decrease by a fixed amount from term to term. For example:

$$1, 3, 5, 7, 9, 11, \dots$$

$$10, 5, 0, -5, -10, \dots$$

$$7.9, 7.8, 7.7, 7.6, \dots$$

A series is derived from a sequence by adding the terms. For example:

$$1 + 3 + 5 + 7 + 9 + 11 + \dots$$

$$10 + 5 + 0 - 5 - 10 - \dots$$

There are two main formulas related to these:

$$\text{The } n\text{th term: } u_n = a + (n-1)d$$

The sum to n terms of the related series:

$$S_n = \frac{n}{2} \times [2a + (n-1)d]$$

Geometric Sequences

These increase or decrease by a fixed ratio from term to term. For example:

$$1, 2, 4, 8, 16, 32, \dots$$

$$1, -1, 1, -1, 1, \dots$$

$$100, 10, 1, 0.1, 0.01, \dots$$

There are two main formulas related to these:

$$\text{The } n\text{th term: } u_n = a \times r^{n-1}$$

The sum to n terms of the related series:

$$S_n = \frac{a(1-r^n)}{1-r}, |r| < 1 \text{ or } S_n = \frac{a(r^n-1)}{r-1}, |r| > 1$$

It might be a good idea to take some time to review the use of these formulas before proceeding.

Convergent series

If a geometric series has a common ratio between -1 and 1 , the terms get smaller and smaller as n increases.

The sum of these terms is still given by the formula

$$S_n = \frac{a(1-r^n)}{1-r}, r \neq 1$$

For $-1 < r < 1$, $r^n \rightarrow 0$ as $n \rightarrow \infty$, $S_n \rightarrow \frac{a}{1-r}$

If $|r| < 1$, the infinite sequence has a sum given by: $S_\infty = \frac{a}{1-r}$

This means that if the common ratio of a geometric series is between -1 and 1 , the sum of the series will approach a value of $\frac{a}{1-r}$ as the number of terms of the series becomes large,

i.e. the series is convergent.

Example A.2.1

Find the sum to infinity of the series:

a $16 + 8 + 4 + 2 + 1 + \dots$

b $9 - 6 + 4 - \frac{8}{3} + \frac{16}{9} + \dots$

a $16 + 8 + 4 + 2 + 1 + \dots$

In this case: a $a = 16, r = \frac{1}{2} \Rightarrow S_\infty = \frac{a}{1-r} = \frac{16}{1-\frac{1}{2}} = 32$

b $9 - 6 + 4 - \frac{8}{3} + \frac{16}{9} + \dots$

$a = 9, r = -\frac{2}{3} \Rightarrow S_\infty = \frac{a}{1-r} = \frac{9}{1-\left(-\frac{2}{3}\right)} = 5.4$

There are many applications for convergent geometric series. The following examples illustrate two of these.

Example A.2.2

Use an infinite series to express the recurring decimal $0.46\bar{2}$ as rational number.

$0.46\bar{2}$ can be expressed as the series:

$$0.462 + 0.000462 + 0.000000462 + \dots$$

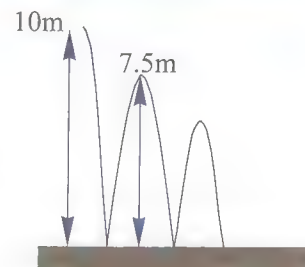
or $\frac{462}{1000} + \frac{462}{1000000} + \frac{462}{1000000000} + \dots$

This is a geometric series with: $a = \frac{462}{1000}, r = \frac{1}{1000}$.

It follows that: $S_\infty = \frac{a}{1-r} = \frac{\frac{462}{1000}}{1-\frac{1}{1000}} = \frac{\frac{462}{1000}}{\frac{999}{1000}} = \frac{462}{999}$

Example A.2.3

A ball is dropped from a height of 10 metres. On each bounce the ball bounces to three quarters of the height of the previous bounce. Find the distance travelled by the ball before it comes to rest (if it does not move sideways).



The ball bounces in a vertical line and does not move sideways. On each bounce after the drop, the ball moves both up and down and so travels twice the distance of the height of the bounce.

$$\text{Distance} = 10 + 15 + 15 \times \frac{3}{4} + 15 \times \left(\frac{3}{4}\right)^2 + \dots$$

All terms, except the first, are geometric.

$$\text{Distance } 10 + S_\infty = 10 + \frac{15}{1-\frac{3}{4}} = 70 \text{ m}$$

Example A.2.4

A unit square is dissected in the manner suggested in the diagram. Find the total shaded area.



Each rectangle has an area of one quarter of the square below and to the left.

The areas form a geometric series: $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots$

$$a = r = \frac{1}{4} \text{ so } S_{\infty} = \frac{a}{1-r} = \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

Exercise A.2.1

1. Evaluate:

a $27 + 9 + 3 + \frac{1}{3} + \dots$

b $1 - \frac{3}{10} + \frac{9}{100} - \frac{27}{1000} + \dots$

c $500 + 450 + 405 + 364.5 + \dots$

d $3 - 0.3 + 0.03 - 0.003 + 0.0003 - \dots$

2. Use geometric series to express the recurring decimal $23.232323\dots$ as a mixed number.
3. Biologists estimate that there are 1000 trout in a lake. If none are caught, the population will increase at 10% per year. If more than 10% are caught, the population will fall. As an approximation, assume that if 25% of the fish are caught per year, the population will fall by 15% per year. Estimate the total catch before the lake is 'fished out'. If the catch rate is reduced to 15%, what is the total catch in this case? Comment on these results.
4. Find the sum to infinity of the sequence $45, -30, 20, \dots$
5. The second term of a geometric sequence is 12 while the sum to infinity is 64. Find the first three terms of this sequence.
6. Express the following as rational numbers:
 - a $0.3\dot{6}$
 - b $0.\dot{3}7$
 - c $2.1\dot{2}$
7. A swinging pendulum covers 32 centimetres in its first swing, 24 cm on its second swing, 18 cm on its third swing and so on. What is the total distance this pendulum swings before coming to rest?
8. The sum to infinity of a geometric sequence is $27/2$ while the sum of the first three terms is 13. Find the sum of the first 5 terms.
9. Find the sum to infinity of the sequence:

$$1 + \sqrt{3}, 1, \frac{1}{\sqrt{3}+1}, \dots$$

10. a Find: i $\sum_{r=0}^n (-t)^r, |t| < 1,$

ii $\sum_{r=0}^{\infty} (-t)^r, |t| < 1.$

- b i Hence, show that ,

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots, |x| < 1$$

- ii Using the above result, show that:

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

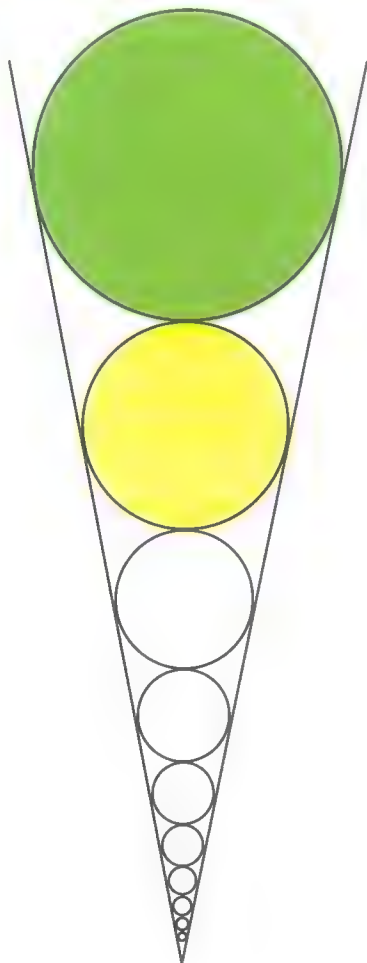
11. a Find: i $\sum_{i=0}^n (-t^2)^i, |t| < 1$, ii $\sum_{i=0}^{\infty} (-t^2)^i, |t| < 1$.

- i Hence, show that:

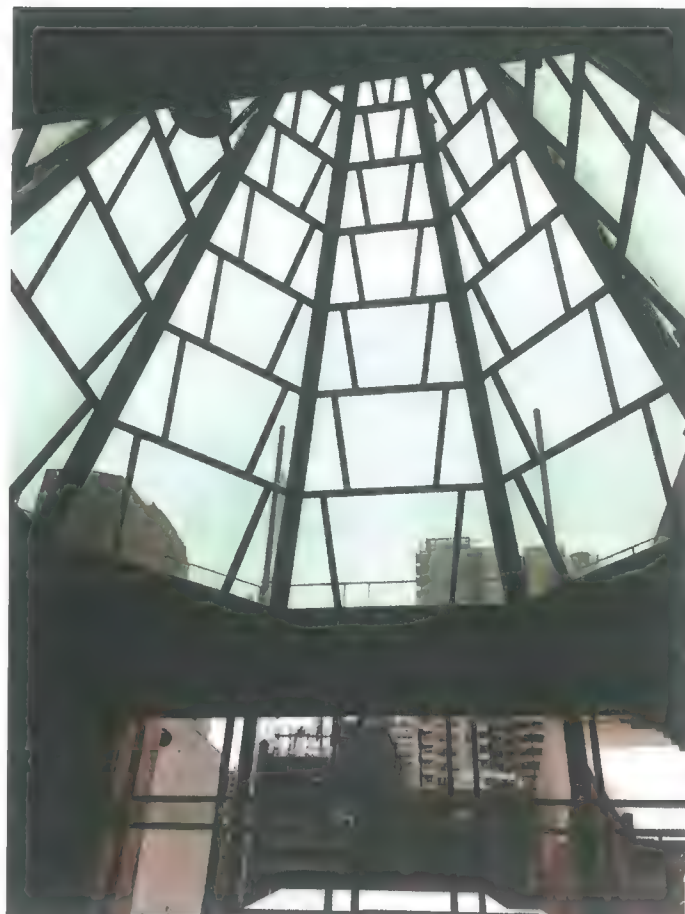
$$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots, |x| < 1$$

Using the above result, show that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$.

12. The green circle has unit area. The yellow circle has a radius two thirds that of the green circle. Find the total area of the infinite series of circles.



13. This photograph suggests a method for using a sequence of trapezia to approximate the area of the curved surface of a cone. How would this work?

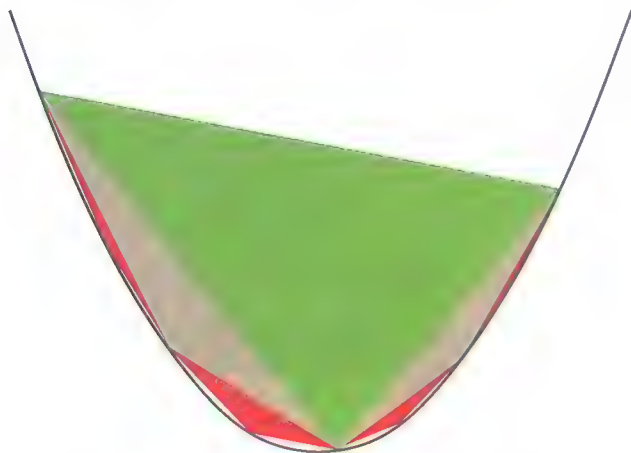




Archimedes and the Parabola

Later in the course, you will use Calculus to compute the areas contained by curves such as the parabola. This method was not developed until the early 1700s. Before then there is a rich history of inventive methods for solving these problems that often involved infinite series.

This diagram suggests the method attributed to Archimedes for finding an area bounded by a parabola.



The dissection is based on the single green triangle. This is followed by 2 brown triangles, 4 red triangles etc. constructed in the manner shown.

Can you find an infinite series that will allow you to relate the total area to that of the original (green) triangle?

Answers



A.3 Exponentials and Logarithms

SL 1.7

Polar Logarithmic Graph

SL1.5 Exponents

Basic rules of indices

We start by looking at the notation involved when dealing with indices (or exponents).

The expression:

$$a \times a \times a \times a \times a \times \dots \times a$$

← n times →

can be written in index form, a^n , where n is the index (or power or exponent) and a is the base.

This expression is read as “ a to the power of n ”, or more briefly as “ a to the n ”. For example, we have that $3^5 = 3 \times 3 \times 3 \times 3 \times 3$ so that 3 is the base and 5 the exponent (or index).

The laws for positive integral indices are summarized below. If a and b are real numbers and m and n are positive integers, we have:

	Law	Rule	Example
1	Multiplication [same base]	$a^m \times a^n = a^{m+n}$	$3^4 \times 3^6 = 3^{4+6} = 3^{10}$
2	Division [same base]	$a^m \div a^n = \frac{a^m}{a^n} = a^{m-n}$	$7^9 \div 7^5 = 7^{9-5} = 7^4$
3	Power of a power [same base]	$(a^m)^n = a^{m \times n}$	$(2^3)^5 = 2^{3 \times 5} = 2^{15}$

	Law	Rule	Example
4	Power of a product	$a^m \times b^m = (ab)^m$	$3^4 \times 7^4 = (3 \times 7)^4 = 21^4$
5	Power of a quotient	$a^m \div b^m = \left(\frac{a}{b}\right)^m$	$5^3 \div 7^3 = \left(\frac{5}{7}\right)^3$
6	Negative one to a power	$(-1)^n = \begin{cases} -1, n \text{ odd} \\ 1, n \text{ even} \end{cases}$	$(-1)^3 = -1$ $(-1)^4 = 1$

There are more laws of indices that are based on rational indices, negative indices and the zero index.

	Law	Rule	Example
1	Fractional index Type 1 [nth root]	$a^{\frac{1}{n}} = \sqrt[n]{a}, n \in \mathbb{N}$ Note: if n is even, then $a \geq 0$. If n is odd, then $a \in \mathbb{R}$	$8^{\frac{1}{3}} = \sqrt[3]{8} = 2$ $(-27)^{\frac{1}{3}} = \sqrt[3]{-27} = -3$
2	Fractional index Type 2	$a^{\frac{m}{n}} = \sqrt[n]{a^m}, n \in \mathbb{N}$	$16^{\frac{3}{4}} = \sqrt[4]{16^3} = 8$
3	Negative index	$a^{-n} = \frac{1}{a^n}, a \neq 0, n \in \mathbb{N}$	$2^{-1} = \frac{1}{2}$ $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$
4	Zero index	$a^0 = 1, a \neq 0$ Note $0^n = 0, a \neq 0$	$12^0 = 1$

We make the following note about fractional indices:

As $\frac{m}{n} = m \times \frac{1}{n} = \frac{1}{n} \times m$, we have that for $b \geq 0$

$$\text{i} \quad b^{\frac{m}{n}} = b^{m \times \frac{1}{n}} = (b^m)^{\frac{1}{n}} = \sqrt[n]{b^m}$$

$$\text{ii} \quad b^{\frac{m}{n}} = b^{\frac{1}{n} \times m} = \left(b^{\frac{1}{n}}\right)^m = (\sqrt[n]{b})^m$$

Then,

$$\text{If } b \geq 0, \text{ then } b^{\frac{m}{n}} = \sqrt[n]{b^m} = (\sqrt[n]{b})^m, m \in \mathbb{Z}, n \in \mathbb{N}$$

$$\text{If } b < 0, \text{ then } b^{\frac{m}{n}} = \sqrt[n]{b^m} = (\sqrt[n]{b})^m, m \in \mathbb{Z}, n \in \{1, 3, 5, \dots\}$$

Example A.3.1

Simplify the following.

$$\text{a} \quad \left(\frac{4x^2}{5y^4}\right)^2 \times (2x^3y)^3 \quad \text{b} \quad \frac{3^{n+1} + 3^2}{3}$$

$$\begin{aligned} \text{a} \quad \left(\frac{4x^2}{5y^4}\right)^2 \times (2x^3y)^3 &= \frac{4^2 x^{2 \times 2}}{5^2 y^{4 \times 2}} \times 2^3 x^{3 \times 3} y^{1 \times 3} \\ &= \frac{16x^4}{25y^8} \times 8x^9y^3 \\ &= \frac{128}{25} x^{4+9} y^{3-8} \\ &= \frac{128}{25} x^{13} y^{-5} \\ &= \frac{128x^{13}}{25y^5} \end{aligned}$$

$$\text{b} \quad \frac{3^{n+1} + 3^2}{3} = \frac{3(3^n + 3)}{3} = 3^n + 3$$

Example A.3.2

Simplify the following:

$$\text{a} \quad \frac{4x^2(-y^{-1})^{-2}}{(-2x^2)^3(y^{-2})^2} \quad \text{b} \quad \frac{x^{-1} + y^{-1}}{x^{-1}y^{-1}}$$

$$\begin{aligned} \text{a} \quad \frac{4x^2(-y^{-1})^{-2}}{(-2x^2)^3(y^{-2})^2} &= \frac{4x^2 \times y^{-1 \times -2}}{-8x^{2 \times 3} \times y^{-2 \times 2}} = \frac{-x^2y^2}{2x^6y^{-4}} \\ &= \frac{-y^2 - (-4)}{2x^6 - (-2)} = \frac{y^6}{2x^4} \end{aligned}$$

$$\text{b} \quad \frac{x^{-1} + y^{-1}}{x^{-1}y^{-1}} = \frac{\frac{1}{x} + \frac{1}{y}}{\frac{1}{xy}} = \left(\frac{1}{x} + \frac{1}{y}\right) \times \frac{xy}{1} = \frac{xy}{x} + \frac{xy}{y} = y + x$$

Example A.3.3

Simplify the following.

$$\text{a} \quad \frac{2^{n-3} \times 8^{n+1}}{2^{2n-1} \times 4^{2-n}} \quad \text{b} \quad \frac{(a^{1/3} \times b^{1/2})^{-6}}{\sqrt[4]{a^8b^9}}$$

$$\begin{aligned} \text{a} \quad \frac{2^{n-3} \times 8^{n+1}}{2^{2n-1} \times 4^{2-n}} &= \frac{2^{n-3} \times (2^3)^{n+1}}{2^{2n-1} \times (2^2)^{2-n}} = \frac{2^{n-3} \times 2^{3n+3}}{2^{2n-1} \times 2^{4-2n}} \\ &= \frac{2^{n-3+(3n+3)}}{2^{2n-1+(4-2n)}} = \frac{2^{4n}}{2^3} = 2^{4n-3} \end{aligned}$$

$$\begin{aligned} \text{b} \quad \frac{(a^{1/3} \times b^{1/2})^{-6}}{\sqrt[4]{a^8b^9}} &= \frac{a^{\frac{1}{3} \times -6} \times b^{\frac{1}{2} \times -6}}{(a^8b^9)^{\frac{1}{4}}} = \frac{a^{-2} \times b^{-3}}{a^2b^{\frac{9}{4}}} \\ &= a^{-2-2} \times b^{-3-\frac{9}{4}} \\ &= a^{-4}b^{-\frac{21}{4}} \\ &= \frac{1}{a^4b^{\frac{21}{4}}} \end{aligned}$$

Exercise A.3.1

1. Simplify the following.

$$\begin{aligned} \text{a} \quad \left(\frac{3y^2}{4x^3}\right)^3 \times (2x^2y^3)^3 & \quad \text{b} \quad \left(\frac{2}{3a^2}\right)^3 + \frac{1}{8a^6} \\ \text{c} \quad \frac{2^{n+1} + 2^2}{2} & \quad \text{d} \quad \left(\frac{2x^3}{3y^2}\right)^3 \times (xy^2)^2 \end{aligned}$$

2. Simplify the following.

$$\begin{aligned} \text{a} \quad \frac{20^6}{10^6} & \quad \text{b} \quad \frac{12^{2x}}{(6^3)^x} \\ \text{c} \quad \frac{16^{2y+1}}{8^{2y+1}} & \quad \text{d} \quad \frac{(ab)^{2x}}{a^{2x}b^{4x}} \end{aligned}$$

3. Simplify the following.

$$\begin{aligned} \text{a} \quad \left(\frac{x}{y}\right)^3 \times \left(\frac{y}{z}\right)^2 \times \left(\frac{z}{x}\right)^4 & \quad \text{b} \quad 3^{2n} \times 27 \times 243^{n-1} \\ \text{c} \quad \frac{25^{2n} \times 5^{1-n}}{(5^2)^n} & \quad \text{d} \quad \frac{9^n \times 3^{n+2}}{27^n} \end{aligned}$$

4. Simplify $\frac{(x^m)^n(y^2)^m}{(x^m)^{(n+1)}y^2}$.
5. Simplify the following, leaving your answer in positive power form.

a $\frac{(-3^4) \times 3^{-2}}{(-3)^{-2}}$ b $\frac{9y^2(-x^{-1})^{-2}}{(-2y^2)^3(x^{-2})^3}$

c $\frac{x^{-1} - y^{-1}}{x^{-1}y^{-1}}$ d $\frac{x^{-2} + 2x^{-1}}{x^{-1} + x^{-2}}$

6. Simplify the following.

a $\frac{(x^{-1})^2 + (y^2)^{-1}}{x^2 + y^2}$ b $\frac{(x^2)^{-2} + 2y}{1 + 2yx^4}$

c $\frac{(x+h)^{-1} - x^{-1}}{h}$

d $(x^2 - 1)^{-1} \times (x + 1)$

Extra questions



Indicial equations

Why do indices turn up so often in applications?

This is a complex question, but it does seem that the Universe has a preference for power laws!

Biologists use power expressions to model animal populations:



Chemists use power laws to track the rates of reactions:



and so on. Is there anything behind this or is nature's liking for indices just chance?

Solving equations of the form $x^{\frac{1}{2}} = 3$, where the **variable is the base**, requires that we square both sides of the equation so that:

$$\left(x^{\frac{1}{2}}\right)^2 = 3^2 \Rightarrow x = 9.$$

However, when the **variable is the power** and not the base we need to take a different approach.

Indicial (exponential) equations take on the general form $b^x = a$, where the unknown variable x is the power.

Consider the case where we wish to solve for x given that $2^x = 8$. In this case we need to think of a value of x so that when 2 is raised to the power of x the answer is 8. Using trial and error, it is not too difficult to arrive at $x = 3$ ($2^3 = 2 \times 2 \times 2 = 8$).

Next consider the equation $3^{x+1} = 27$. Again, we need to find a number such that when 3 is raised to that number, the answer is 27. Here we have that $27 = 3^3$. Therefore we can rewrite the equation as $3^{x+1} = 3^3$.

As the base on both sides of the equality is the same we can then equate the powers, that is,

$$3^{x+1} = 27 \Leftrightarrow 3^{x+1} = 3^3$$

$$\Leftrightarrow x + 1 = 3$$

$$\Leftrightarrow x = 2$$

Example A.3.4

Solve the following:

a $3^x = 81$ b $2 \times 5^u = 250$ c $2^x = \frac{1}{32}$

a $3^x = 81 \Leftrightarrow 3^x = 3^4 \Leftrightarrow x = 4$

b $2 \times 5^u = 250 \Leftrightarrow 5^u = 125$

$$\Leftrightarrow 5^u = 5^3$$

$$\Leftrightarrow u = 3$$

c $2^x = \frac{1}{32} \Leftrightarrow 2^x = \frac{1}{2^5}$

$$\Leftrightarrow 2^x = 2^{-5}$$

$$\Leftrightarrow x = -5$$

Example A.3.5

Find:

a $\left\{x \mid \left(\frac{1}{2}\right)^x = 16\right\}$ b $\{x \mid 3^{x+1} = 3\sqrt{3}\}$

a $\left(\frac{1}{2}\right)^x = 16 \Leftrightarrow (2^{-1})^x = 16$
 $\Leftrightarrow 2^{-x} = 2^4$
 $\Leftrightarrow -x = 4$
 $\Leftrightarrow x = -4$

i.e. solution set is $\{-4\}$.

b $3^{x+1} = 3\sqrt{3} \Leftrightarrow 3^{x+1} = 3 \times 3^{1/2}$
 $\Leftrightarrow 3^{x+1} = 3^{3/2}$
 $\Leftrightarrow x+1 = \frac{3}{2}$
 $\Leftrightarrow x = \frac{1}{2}$

i.e. solution set is $\{0.5\}$

Exercise A.3.2

1. Solve the following equations.

a $\{x \mid 4^x = 16\}$ b $\left\{x \mid 7^x = \frac{1}{49}\right\}$
c $\{x \mid 8^x = 4\}$ d $\{x \mid 3^x = 243\}$
e $\{x \mid 3^{x-2} = 81\}$ f $\left\{x \mid 4^x = \frac{1}{32}\right\}$

2. Solve the following equations.

a $\{x \mid 7^{x+6} = 1\}$ b $\left\{x \mid 8^x = \frac{1}{4}\right\}$
c $\{x \mid 10^x = 0.001\}$ d $\{x \mid 9^x = 27\}$
e $\{x \mid 2^{4x-1} = 1\}$ f $\{x \mid 25^x = \sqrt{5}\}$
g $\left\{x \mid 16^x = \frac{1}{\sqrt{2}}\right\}$ h $\{x \mid 4^{-x} = 32\sqrt{2}\}$
i $\{x \mid 9^{-2x} = 243\}$

3. Show that: $\frac{1}{16}(8^{3x+2}) = 2^{9x+2}$

Extra questions



Drag is the force resisting the forward motion of vehicles such as aircraft. It is a 'square law' - the drag force is proportional to the square of the velocity.



LOGARITHMS

What are logarithms?

Consider the following sequence of numbers:

Sequence N	1	2	3	4	5	6	7	8	9	10	...
Sequence y	2	4	8	16	32	64	128	256	512	1024	...

The relationship between the values of N and y is given by $y = 2^N$.

We can use the above table, to evaluate the product 16×64 .

Sequence N	1	2	3	4	5	6	7	8	9	10	...
Sequence y	2	4	8	16	32	64	128	256	512	1024	...

We start by setting up a table of values that correspond to the numbers in question:

N	4	6	Sum(4 + 6) = 10
y	16	64	Product = 1024

From the first table of sequences, we notice that the sum of the 'N sequence' (i.e. 10), corresponds to the value of the 'y sequence' (i.e. 1024).

We next consider the product 16×64 , again. Setting up a table of values for the numbers in the sequences that are under investigation we have:

N	3	5	$\text{Sum}(3 + 5) = 8$
y	8	32	$\text{Product} = 256$

What about 4×64 ? As before, we set up the required table of values:

N	2	6	$\text{Sum}(2 + 6) = 8$
y	4	64	$\text{Product} = 256$

In each case the **product** of two terms of the sequence y corresponds to the **sum** obtained by adding corresponding terms of the sequence N .

Notice that **dividing** two numbers from the sequence y corresponds to the result when **subtracting** the two corresponding numbers from the sequence N ,

e.g. for the sequence y : $512 \div 32 = 16$.

for the sequence N : $9 - 5 = 4$

This remarkable property was observed as early as 1594 by **John Napier**. John Napier was born in 1550 (when his father was all of sixteen years of age!) He lived most of his life at the family estate of Merchiston Castle, near Edinburgh, Scotland. Although his life was not without controversy, in matters both religious and political, Napier (when relaxing from his political and religious polemics) would indulge in the study of mathematics and science. His amusement with the study of mathematics led him to the invention of logarithms. In 1614 Napier published his discussion of logarithms in a brochure entitled *Mirifici logarithmorum canonis descriptio* ('A description of the Wonderful Law of Logarithms'). Napier died in 1617.



It is only fair to mention that the Swiss instrument maker **Jobst Bürgi** (1552–1632) conceived and constructed a table of logarithms independently of Napier, publishing his results in 1620, six years after Napier had announced his discovery.

One of the anomalies in the history of mathematics is the fact that logarithms were discovered before exponents were in use.

In this age of technology, the use of electronic calculators and computers has reduced the evaluation of products and quotients to tasks that involve the simple push of a

few buttons. However, logarithms are an efficient means of converting a product to a sum and a quotient to a difference. So, what are logarithms?

Nowadays, a logarithm is universally regarded as an exponent.

Thus, if $y = b^x$ we say that x is the logarithm of y to the base b .

From the sequence table, we have that $2^7 = 128$, so that 7 is the **logarithm** of 128 to the base 2.

Similarly, $3^4 = 81$, and so 4 is the **logarithm** of 81 to the base 3.

$y = b^x$ we say that $x = \log_b y$.

That is, N is the logarithm of y to the base b , which corresponds to the power that the base b must be raised so that the result is y .

To determine the number $\log_2 32$, we ask ourselves the following question: "To what power must we raise the number 2, so that our result is 32?"

Letting $x = \log_2 32$, we must find the number x such that $2^x = 32$.

Clearly then, $x = 5$, and so we have that $\log_2 32 = 5$.

One convention in setting out such questions is:

$$\log_2 32 = x \Leftrightarrow 2^x = 32$$

$$\Leftrightarrow x = 5$$

As in part a, we ask the question "To what power must we raise the number 10, so that our result is 1000?"

$$\text{That is, } x = \log_{10} 1000 \Leftrightarrow 10^x = 1000$$

$$\Leftrightarrow x = 3$$

$$\text{So that } \log_{10} 1000 = 3.$$

$$\text{Now, } x = \log_3 729 \Leftrightarrow 3^x = 729 \Leftrightarrow x = 6$$

(This was obtained by trial and error.)

$$\text{Therefore, } \log_3 729 = 6.$$

Although we have a fraction, this does not alter the process:

$$x = \log_2 \frac{1}{16} \Leftrightarrow 2^x = \frac{1}{16}$$

$$\Leftrightarrow 2^x = \frac{1}{2^4} (= 2^{-4})$$

$$\Leftrightarrow x = -4$$

Can we find the logarithm of a negative number?

To evaluate $\log_a(-4)$ for some base $a > 0$, we need to solve the equivalent statement:

$$x = \log_a(-4) \Leftrightarrow a^x = -4$$

However, the value of a^x where $a > 0$, will always be positive, therefore there is no value of x for which $a^x = -4$. This means that



We can now make our definition a little stronger:

$$N = \log_b y \Leftrightarrow y = b^N, y > 0$$

Example A.3.6

Find the value of x given that:

$$\text{a } \log_e x = 3 \quad \text{b } \log_e(x-2) = 0.5 \quad \text{c } \log_e 5 = x$$

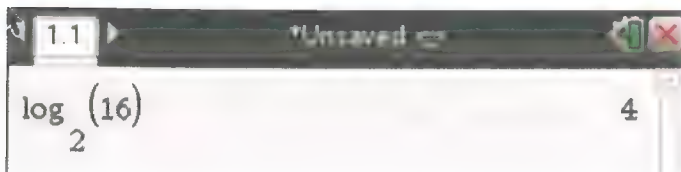
$$\text{a } \log_e x = 3 \Leftrightarrow x = e^3 \approx 20.09$$

$$\text{b } \log_e(x-2) = 0.5 \Leftrightarrow x-2 = e^{0.5}$$

$$\Leftrightarrow x = 2 + \sqrt{e}$$

$$\therefore x \approx 3.65$$

$$\text{c } x = \log_e 5 \approx 1.61$$



The Nautilus

The Nautilus is a very ancient (and we can infer, successful) sea creature.

This is the fossil of an ammonite that appears to have been a precursor of the Nautilus. Note the logarithmic spiral shape.



The different sizes of the segments probably result from the natural growth of the animal.

A nautilus in action:

It is jet-propelled.



Exercise A.3.3

- Use the definition of a logarithm to determine the following.

$$\text{a } \log_6 36 \quad \text{b } \log_7 49 \quad \text{c } \log_3 243$$

$$\text{d } \log_4 64 \quad \text{e } \log_2 \left(\frac{1}{8}\right) \quad \text{f } \log_3 \left(\frac{1}{9}\right)$$

- Change the following exponential expressions into their equivalent logarithmic form.

$$\text{a } 10^4 = 10000 \quad \text{b } 10^{-3} = 0.001$$

$$\text{c } 10^y = x + 1 \quad \text{d } 10^7 = p$$

$$\text{e } 2^y = x - 1 \quad \text{f } 2^{4x} = y - 2$$

- Change the following logarithmic expressions into their equivalent exponential form.

$$\text{a } \log_2 x = 9 \quad \text{b } \log_b y = x \quad \text{c } \log_b t = ax$$

$$\text{d } \log_{10} z = x^2 \quad \text{e } \log_{10} y = 1 - x$$

- Solve for x in each of the following.

$$\text{a } \log_2 x = 4 \quad \text{b } \log_3 9 = x \quad \text{c } \log_4 x = \frac{1}{2}$$

$$\text{d } \log_x 3 = \frac{1}{2} \quad \text{e } \log_x 2 = 4 \quad \text{f } \log_5 x = 3$$

5. Solve for x in each of the following, giving your answer to 4 d.p.

$$a \log_e x = 4 \quad b \log_e 4 = x \quad c \log_e x = \frac{1}{2}$$

$$d \log_x e = \frac{1}{2} \quad e \log_x e = 2 \quad f \log_x e = -1$$

Extra questions



THE ALGEBRA OF LOGARITHMS

The following logarithmic laws are a direct consequence of the definition of a logarithm and the index laws already established.

First law: The logarithm of a product

$$\log_a(x \times y) = \log_a x + \log_a y, x > 0, y > 0$$

Proof: Let $M = \log_a x$ and $N = \log_a y$ so that $x = a^M$ and $y = a^N$.

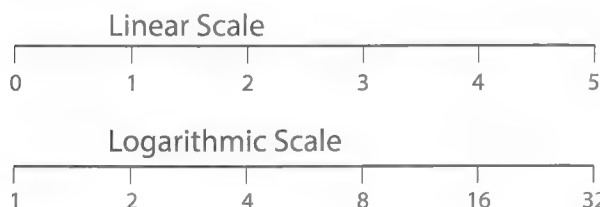
$$\text{Then, } x \times y = a^M \times a^N$$

$$\Leftrightarrow x \times y = a^{M+N}$$

$$\Leftrightarrow \log_a(x \times y) = M + N$$

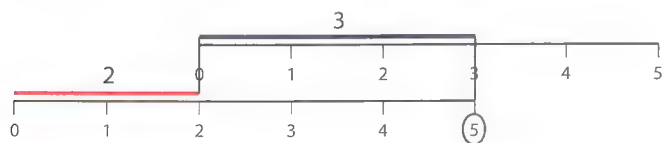
$$\Leftrightarrow \log_a(x \times y) = \log_a x + \log_a y$$

Originally, logarithms were viewed as a series. This led to the logarithmic scale.

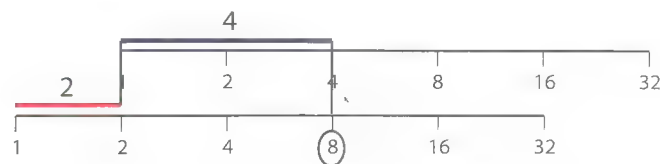


The linear scale is arithmetic and the logarithmic scale is geometric.

A pair of linear scales can be used to add numbers: In this case $2 + 3 = 5$.



A pair of logarithmic scales can be used to multiply numbers. In this case $2 \times 4 = 8$



Note that other multiplications by 2 can be read from the diagram.

This is a brief demonstration of the use of a slide rule:



Example A.3.7

Simplify the following expressions:

$$a \log_3 x + \log_3(4x) \quad b \log_2 x + \log_2(4x)$$

$$\begin{aligned} a \quad \log_3 x + \log_3(4x) &= \log_3(x \times 4x) \\ &= \log_3 4x^2 \end{aligned}$$

$$\begin{aligned} b \quad \text{This time we note that because the base is '2' and there is a '4' in one of the logarithmic expressions, we could first try to remove the '4'.} \\ \log_2 x + \log_2(4x) &= \log_2 x + (\log_2 4 + \log_2 x) \\ &= \log_2 x + 2 + \log_2 x \\ &= 2\log_2 x + 2 \end{aligned}$$

Example A.3.8

Given that $\log_a p = 0.70$ and $\log_a q = 2$, evaluate the following.

$$a \log_a p^2 \quad b \log_a(p^2 q) \quad c \log_a(apq)$$

$$\begin{aligned} a \quad \log_a p^2 &= \log_a(p \times p) = \log_a p + \log_a p \\ &= 2\log_a p \\ &= 2 \times 0.70 \\ &= 1.40 \end{aligned}$$

$$\begin{aligned} \text{b} \quad \log_a(p^2q) &= \log_a p^2 + \log_a q \\ &= 2\log_a p + \log_a q \\ &= 1.40 + 2 \\ &= 3.40 \end{aligned}$$

$$\begin{aligned} \text{c} \quad \log_a(apq) &= \log_a a + \log_a p + \log_a q \\ &= 1 + 0.70 + 2 \\ &= 3.70 \end{aligned}$$

Example A.3.9

Find $\{x \mid \log_2 x + \log_2(x+2) = 3\}$.

$$\begin{aligned} \log_2 x + \log_2(x+2) &= 3 \Leftrightarrow \log_2[x \times (x+2)] = 3 \\ \Leftrightarrow x(x+2) &= 2^3 \\ \Leftrightarrow x^2 + 2x &= 8 \\ \Leftrightarrow x^2 + 2x - 8 &= 0 \\ \Leftrightarrow (x+4)(x-2) &= 0 \\ \Leftrightarrow x &= -4 \text{ or } x = 2 \end{aligned}$$

Next, we must check our solutions.

When $x = -4$, substituting into the **original equation**, we have: L.H.S. $= \log_2(-4) + \log_2(-4+2)$ – which cannot be evaluated (as the logarithm of a negative number does not exist). Therefore, $x = -4$, is not a possible solution.

When $x = 2$, substituting into the **original equation**, we have:

$$\begin{aligned} \text{L.H.S.} &= \log_2(2) + \log_2(2+2) \\ &= \log_2 8 \\ &= 3 \\ &= \text{R.H.S.} \end{aligned}$$

Therefore, $\{x \mid \log_2 x + \log_2(x+2) = 3\} = \{2\}$.

Second law: The logarithm of a quotient

$$\log \left(\frac{x}{y} \right) = \log x - \log y, \quad x > 0, y > 0$$

Proof: Let $M = \log_a x$ and $N = \log_a y$ so that $x = a^M$

and $y = a^N$.

$$\text{Then, } \frac{x}{y} = \frac{a^M}{a^N}$$

$$\Leftrightarrow \frac{x}{y} = a^{M-N}$$

$$\Leftrightarrow \log_a \left(\frac{x}{y} \right) = M - N$$

$$\Leftrightarrow \log_a \left(\frac{x}{y} \right) = \log_a x - \log_a y$$

Example A.3.10

Simplify: a $\log_{10} 100x - \log_{10} xy$

$$\text{b } \log_2 8x^3 - \log_2 x^2 + \log_2 \left(\frac{y}{x} \right)$$

$$\begin{aligned} \text{a} \quad \log_{10} 100x - \log_{10} xy &= \log_{10} \left(\frac{100x}{xy} \right) \\ &= \log_{10} \left(\frac{100}{y} \right) \end{aligned}$$

Note: We could then express $\log_{10} \left(\frac{100}{y} \right)$ as:

$$\log_{10} 100 - \log_{10} y = 2 - \log_{10} y.$$

$$\begin{aligned} \text{b} \quad \log_2 8x^3 - \log_2 x^2 + \log_2 \left(\frac{y}{x} \right) &= \log_2 \left(\frac{8x^3}{x^2} \right) + \log_2 \left(\frac{y}{x} \right) \\ &= \log_2 8x + \log_2 \left(\frac{y}{x} \right) \\ &= \log_2 \left(8x \times \frac{y}{x} \right) \\ &= \log_2 8y \end{aligned}$$

Note: We could then express $\log_2 8y$ as:

$$\log_2 8 + \log_2 y = 3 + \log_2 y.$$

Example A.3.11

Find $\{x \mid \log_{10}(x+2) - \log_{10}(x-1) = 1\}$.

$$\log_{10}(x+2) - \log_{10}(x-1) = 1 \Leftrightarrow \log_{10} \left(\frac{x+2}{x-1} \right) = 1$$

$$\Leftrightarrow \left(\frac{x+2}{x-1}\right) = 10^1$$

$$\Leftrightarrow x+2 = 10x-10$$

$$\Leftrightarrow 12 = 9x$$

$$\Leftrightarrow x = \frac{4}{3}$$

Next, we check our answer. Substituting into the original equation, we have:

L.H.S

$$\begin{aligned} &= \log_{10} \left(\frac{4}{3} + 2\right) - \log_{10} \left(\frac{4}{3} - 1\right) = \log_{10} \frac{10}{3} - \log_{10} \frac{1}{3} = \log_{10} \left(\frac{10}{3} \div \frac{1}{3}\right) \\ &= \log_{10} 10 = 1 = \text{R.H.S} \end{aligned}$$

$$\text{Therefore, } \{x \mid \log_{10}(x+2) - \log_{10}(x-1) = 1\} = \left\{\frac{4}{3}\right\}$$

Third law: The logarithm of a power

$$\log_a x^n = n \log_a x, x > 0$$

Proof: This follows from repeated use of the First Law or it can be shown as follows:

$$\text{Let } M = \log_a x \Leftrightarrow a^M = x \Leftrightarrow (a^M)^n = x^n$$

(Raising both sides to the power of n)

$$\Leftrightarrow a^{nM} = x^n$$

(Using the index laws)

$$\Leftrightarrow nM = \log_a x^n$$

(Converting from exponential to log form)

$$\Leftrightarrow n \log_a x = \log_a x^n$$

Example A.3.12

Given that $\log_a x = 0.2$ and $\log_a y = 0.5$, evaluate:

a $\log_a x^3 y^2$

b $\log_a \sqrt{\frac{x}{y^4}}$

$$\begin{aligned} \text{a } \log_a x^3 y^2 &= \log_a x^3 + \log_a y^2 \\ &= 3 \log_a x + 2 \log_a y \\ &= 3 \times 0.2 + 2 \times 0.5 \\ &= 1.6 \end{aligned}$$

$$\begin{aligned} \text{b } \log_a \sqrt{\frac{x}{y^4}} &= \log_a \left(\frac{x}{y^4}\right)^{1/2} = \frac{1}{2} \log_a \left(\frac{x}{y^4}\right) \\ &= \frac{1}{2} [\log_a (x) - \log_a y^4] \\ &= \frac{1}{2} [\log_a x - 4 \log_a y] \\ &= \frac{1}{2} [0.2 - 4 \times 0.5] \\ &= -0.9 \end{aligned}$$

Fourth law: Change of base

$$\log_a b = \frac{\log_k b}{\log_k a}, a, b, k > 0, k \neq 1$$

Proof: Let $\log_a b = N$ so that $a^N = b$

Taking the logarithms to base k of both sides of the equation we have:

$$\begin{aligned} \log_k (a^N) &= \log_k b \Leftrightarrow N \log_k a = \log_k b \\ \Leftrightarrow N &= \frac{\log_k b}{\log_k a} \end{aligned}$$

However, we have that $\log_a b = N$, therefore, $\log_a b = \frac{\log_k b}{\log_k a}$

Other observations include:

$$\begin{aligned} 1. \log_a a &= 1 \\ 2. \log_a 1 &= 0 \\ 3. \log_a x &= -\log_a \frac{1}{x}, x > 0 \\ 4. \log_a \frac{1}{x} &= -\log_a x \\ 5. a^{\log_a x} &= x, x > 0 \end{aligned}$$

MISCELLANEOUS EXAMPLES

Example A.3.13

Express y in terms of x if:

a $2 + \log_{10} x = 4 \log_{10} y$

b $\log x = \log(a - by) - \log a$

a Given that $2 + \log_{10} x = 4 \log_{10} y$

then $2 = 4 \log_{10} y - \log_{10} x$

$$\Leftrightarrow 2 = \log_{10} y^4 - \log_{10} x$$

$$\Leftrightarrow 2 = \log_{10} \left(\frac{y^4}{x} \right)$$

$$\Leftrightarrow 10^2 = \frac{y^4}{x}$$

$$\Leftrightarrow y^4 = 100x$$

$$\Leftrightarrow y = \sqrt[4]{100x} \quad (\text{as } y > 0)$$

b Given that $\log x = \log(a - by) - \log a$

then $\log x = \log \frac{a - by}{a}$

$$\Leftrightarrow x = \frac{a - by}{a}$$

$$\Leftrightarrow ax = a - by$$

$$\Leftrightarrow by = a - ax$$

$$\Leftrightarrow y = \frac{a}{b}(1 - x)$$

Example A.3.14

Find x if:

a $\log_x 64 = 3$ b $\log_{10} x - \log_{10}(x - 2) = 1$

a $\log_x 64 = 3 \Leftrightarrow x^3 = 64$

$$\Leftrightarrow x^3 = 4^3$$

$$\Leftrightarrow x = 4$$

b $\log_{10} x - \log_{10}(x - 2) = 1 \Leftrightarrow \log_{10} \left(\frac{x}{x - 2} \right) = 1$

$$\Leftrightarrow \frac{x}{x - 2} = 10^1$$

$$\Leftrightarrow x = 10x - 20$$

$$\Leftrightarrow -9x = -20$$

$$\Leftrightarrow x = \frac{20}{9}$$

Check:

Calculator window showing: $\frac{20}{9} \rightarrow x$, $\log_{10} \left(\frac{x}{x-2} \right) = 1$

Example A.3.15

Find $\{x \mid 5^x = 2^{x+1}\}$. Give both an exact answer and one to 2 d.p.

Taking the logarithm of base 10 of both sides $5^x = 2^{x+1}$ gives:

$$5^x = 2^{x+1} \Leftrightarrow \log_{10} 5^x = \log_{10} 2^{x+1}$$

$$\Leftrightarrow x \log_{10} 5 = (x + 1) \log_{10} 2$$

$$\Leftrightarrow x \log_{10} 5 - x \log_{10} 2 = \log_{10} 2$$

$$\Leftrightarrow x(\log_{10} 5 - \log_{10} 2) = \log_{10} 2$$

$$\Leftrightarrow x = \frac{\log_{10} 2}{\log_{10} 5 - \log_{10} 2}$$

And so, $x = 0.75647... = 0.76$ (to 2 d.p.).

Exact answer = $\left\{ \frac{\log_{10} 2}{\log_{10} 5 - \log_{10} 2} \right\}$, answer to 2 d.p. = $\{0.76\}$

Example A.3.16

Find x where $6e^{2x} - 17 \times e^x + 12 = 0$.

We first note that $6e^{2x} - 17 \times e^x + 12$ can be written as $6 \times e^{2x} - 17 \times e^x + 12$.

This in turn can be expressed as $6 \times (e^x)^2 - 17 \times e^x + 12$.

Therefore, making the substitution $y = e^x$, we have that $6 \times (e^x)^2 - 17 \times e^x + 12 = 6y^2 - 17y + 12$ (i.e. we have a 'hidden' quadratic).

Solving for y , we have:

$$6y^2 - 17y + 12 = 0 \Leftrightarrow (2y - 3)(3y - 4) = 0$$

So that $y = \frac{3}{2}$ or $y = \frac{4}{3}$

However, we wish to solve for x , and so we need to substitute back:

$$e^x = \frac{3}{2} \text{ or } e^x = \frac{4}{3}$$

$$\Leftrightarrow x = \ln \frac{3}{2} \text{ or } x = \ln \frac{4}{3}$$

Example A.3.17Solve for x where $8^{2x+1} = 4^{5-x}$.

Taking logs of both sides of the equation $8^{2x+1} = 4^{5-x}$, we have

$$\log 8^{2x+1} = \log 4^{5-x} \Leftrightarrow (2x+1)\log 8 = (5-x)\log 4$$

$$\Leftrightarrow (2x+1)\log 2^3 = (5-x)\log 2^2$$

$$\Leftrightarrow 3(2x+1)\log 2 = 2(5-x)\log 2$$

Therefore, we have that $6x+3 = 10-2x \Leftrightarrow 8x = 7$

$$\therefore x = \frac{7}{8}$$

Exercise A.3.4

1. Without using a calculator, evaluate the following.

a $\log_2 8 + \log_2 4$

b $\log_6 18 + \log_6 2$

c $\log_5 2 + \log_5 12.5$

d $\log_3 18 - \log_3 6$

2. Write down an expression for $\log a$ in terms of $\log b$ and $\log c$ for the following.

a $a = bc$

b $a = b^2c$

c $a = \frac{1}{c^2}$

d $a = b\sqrt{c}$

3. Given that $\log_a x = 0.09$, find:

a $\log_a x^2$

b $\log_a \sqrt{x}$

c $\log_a \left(\frac{1}{x}\right)$

4. Express each of the following as an equation that does not involve a logarithm.

a $\log_2 x = \log_2 y + \log_2 z$

b $\log_{10} y = 2\log_{10} x$

c $\log_2(x+1) = \log_2 y + \log_2 x$

5. Solve the following equations.

a $\log_2(x+1) - \log_2 x = \log_2 3$

b $\log_{10}(x+1) - \log_{10} x = \log_{10} 3$

c $\log_2(x+1) - \log_2(x-1) = 4$

6. Simplify the following

a $\log_3(2x) + \log_3 w$

b $\log_4 x - \log_4(7y)$

7. Solve the following

a $\log_2(x+7) + \log_2 x = 3$

b $\log_3(x+3) + \log_3(x+5) = 1$

c $\log_{10}(x+7) + \log_{10}(x-2) = 1$

8. Solve for x .

a $\log_2 x^2 = (\log_2 x)^2$

b $\log_3 x^3 = (\log_3 x)^3$

9. Solve the following, giving an exact answer and an answer to 2 d.p.

a $2^x = 14$

b $10^x = 8$

c $3^x = 125$

d $\frac{1}{1-2^x} = 12$

10. Solve for x .

a $(\log_2 x)^2 - \log_2 x - 2 = 0$

b $\log_2(2^{x+1} - 8) = x$

11. Solve the following simultaneous equations.

Extra questions



a
$$\begin{aligned} x^y &= 5x - 9 \\ \log_x 11 &= y \end{aligned}$$

b
$$\begin{aligned} \log_{10} x - \log_{10} y &= 1 \\ x + y^2 &= 200 \end{aligned}$$

12. Express each of the following as an equation that does not involve a logarithm.

Answers



a $\log_e x = \log_e y - \log_e z$

b $3\log_e x = \log_e y$

13. Solve the following for x .

a $\ln(x+1) - \ln x = 4$

b $\ln(x+1) - \ln x = \ln 4$

14. Solve the following for x .

a $e^x = 21$

b $e^x - 2 = 8$

15. Solve the following for x .

a $e^{2x} - 3e^x + 2 = 0$

b $e^{2x} - 4e^x - 5 = 0$

16. If: $\log_k(x+3) = \log_k x - 4$, prove that:

A.4 Binomial Theorem

SL 1.9

The Binomial Theorem

The binomial theorem is an example of the use of combinatorics.

Bracketed expressions such as $(2x-3)^7$ are said to be 'binomial' because there are two terms in the bracket (the prefix *bi* means two). Such expressions can be expanded using the distributive law. In a simple case such as $(a+b)^2$ the distributive law gives:

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b) \\ &= a^2 + ab + ba + b^2 \\ &= a^2 + 2ab + b^2\end{aligned}$$

The distributive law states that each term in the first bracket must be multiplied by each term in the second bracket.

The next most complicated binomial can be evaluated using the previous result:

$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)^2 \\ &= (a+b)(a^2 + 2ab + b^2) \\ &= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3 \\ &= a^3 + 3a^2b + 3ab^2 + b^3\end{aligned}$$

Similarly, the fourth power of this simple binomial expression can be expanded as:

$$\begin{aligned}(a+b)^4 &= (a+b)(a+b)^3 \\ &= (a+b)(a^3 + 3a^2b + 3ab^2 + b^3) \\ &= a^4 + 3a^3b + 3a^2b^2 + ab^3 + a^3b + 3a^2b^2 + 3ab^3 + b^4 \\ &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4\end{aligned}$$

The calculations are already fairly complex and it is worth looking at these results for the underlying pattern. There are three main features to the pattern. Looking at the fourth power example above, these patterns are:

The powers of a.

These start at 4 and decrease: a^4, a^3, a^2, a^1, a^0 . Remember that $a^0 = 1$

The powers of b.

These start at 0 and increase: b^0, b^1, b^2, b^3, b^4 . Putting these two patterns together gives the final pattern of terms in which the sum of the indices is always 4:

$$\begin{array}{ccccccc} & & 4 & & 3+1=4 & & 2+2=4 & & 1+3=4 & & 4 \\ a^4 & + & \dots & a^3b^1 & + & \dots & a^2b^2 & + & \dots & a^1b^3 & + & \dots & b^4\end{array}$$

The coefficients complete the pattern.

These coefficients arise because there is more than one way of producing most of the terms. Following the pattern begun above produces a triangular pattern of coefficients known as Pascal's Triangle.

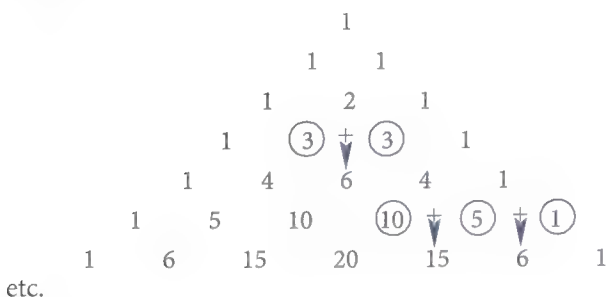
Blaise Pascal (1623-1662) developed early probability theory but is lucky to have this triangle named after him as it had been studied by Chinese mathematicians long before he was born.



So, if we continue our expansions (up to and including the sixth power) we have the following:

$$\begin{aligned}
 (x+a)^0 &= 1 \\
 (x+a)^1 &= 1x + 1a \\
 (x+a)^2 &= 1x^2 + 2ax + 1a^2 \\
 (x+a)^3 &= 1x^3 + 3x^2a + 3xa^2 + 1a^3 \\
 (x+a)^4 &= 1x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + 1a^4 \\
 (x+a)^5 &= 1x^5 + 5x^4a + 10x^3a^2 + 10x^2a^3 + 5xa^4 + 1a^5 \\
 (x+a)^6 &= 1x^6 + 6x^5a + 15x^4a^2 + 20x^3a^3 + 15x^2a^4 + 6xa^5 + 1a^6 \\
 &\vdots \\
 \text{etc.} \quad \text{etc.} &\quad \text{etc.}
 \end{aligned}$$

Now consider only the coefficients for the above expansions. Writing down these coefficients we reproduce Pascal's triangle:



The numbers in the body of the triangle are found by adding the two numbers immediately above and to either side.

An alternative to using Pascal's Triangle to find the coefficients is to use combinatorial numbers. If expanding $(a+b)^5$ the set of coefficients are:

$$\binom{5}{0} = 1, \binom{5}{1} = 5, \binom{5}{2} = 10, \binom{5}{3} = 10, \binom{5}{4} = 5, \binom{5}{5} = 1$$

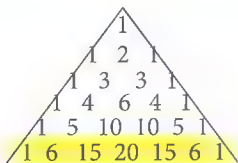
which are the same as those given by Pascal's Triangle.

Most calculators can do this.

Example A.4.1

Expand $(x+y)^6$.

Making use of Pascal's triangle we first determine the required coefficients:



Write down each term:

$$x^6y^0 \quad x^5y^1 \quad x^4y^2 \quad x^3y^3 \quad x^2y^4 \quad x^1y^5 \quad x^0y^6$$

Combine the two steps:

$$x^6y^0 + 6x^5y^1 + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6x^1y^5 + x^0y^6$$

We can also combine the two steps into one:

$$\begin{aligned}
 (x+y)^6 &= {}^6C_0x^6y^0 + {}^6C_1x^5y^1 + {}^6C_2x^4y^2 + {}^6C_3x^3y^3 + \dots \\
 &\quad \dots + {}^6C_4x^2y^4 + {}^6C_5x^1y^5 + {}^6C_6x^0y^6 \\
 &= x^6y^0 + 6x^5y^1 + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6x^1y^5 + x^0y^6
 \end{aligned}$$

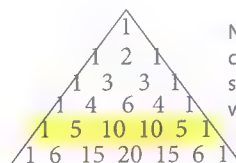
With practice, you should be able to expand such expressions as above.

What happens if we have the difference of two terms rather than the sum? For example, what about expanding $(x-a)^5$? The process is the same except that this time we rewrite the expression $(x-a)^5$ as follows:

$$(x-a)^5 = (x+(-a))^5$$

So, how do we proceed? Essentially, in exactly the same way.

Making use of Pascal's triangle we first determine the required coefficients:



Notice that Pascal's triangle does not change due to the change in sign and so the coefficients remain the same whether it is a sum or difference.

Write down each term:

$$\begin{aligned}
 (x)^5(a)^0 \quad (x)^4(-a) \quad (x)^3(-a)^2 \quad (x)^2(-a)^3 \\
 (x)^1(-a)^4 \quad (x)^0(-a)^5
 \end{aligned}$$

$$\text{i.e.} \quad x^5 \quad -x^4a \quad x^3a^2 \quad -x^2a^3 \quad xa^4 \quad -a^5$$

Combining the coefficients and the terms we have:

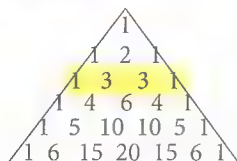
$$(x-a)^5 = x^5 - 5x^4a + 10x^3a^2 - 10x^2a^3 + 5xa^4 - a^5$$

Example A.4.2

Expand

$$\text{a } (4x-3)^3 \quad \text{b } \left(2x - \frac{2}{x}\right)^3$$

- a Making use of Pascal's triangle we first determine the required coefficients:



Write down each term:

$$(4x)^3 \quad (4x)^2(-3) \quad (4x)(-3)^2 \quad (-3)^3$$

$$64x^3 \quad -48x^2 \quad 36x \quad -27$$

Combining steps 1 and 2 we have:

$$\begin{aligned} (4x-3)^3 &= 1 \times 64x^3 + 3 \times -48x^2 + 3 \times 36x + 1 \times -27 \\ &= 64x^3 - 144x^2 + 108x - 27 \end{aligned}$$

- b As with (a), the term pattern must be built on $(2x)$ and $-\frac{2}{x}$:

$$\begin{aligned} \left(2x - \frac{2}{x}\right)^3 &= 1 \times (2x)^3 \left(-\frac{2}{x}\right)^0 + 3 \times (2x)^2 \left(-\frac{2}{x}\right)^1 + \dots \\ &\quad \dots + 3 \times (2x)^1 \left(-\frac{2}{x}\right)^2 + 1 \times (2x)^0 \left(-\frac{2}{x}\right)^3 \\ &= 8x^3 - 24x + \frac{24}{x} - \frac{8}{x^3} \end{aligned}$$

Exercise A.4.1

1. Expand the following binomial expressions:

a $(b+c)^2$

b $(a+g)^3$

c $(1+y)^3$

d $(2+x)^4$

e $(2+2x)^3$

f $(2x-4)^3$

g $\left(2 + \frac{x}{7}\right)^4$

h $(2x-5)^3$

i $(3x-4)^3$

j $(3x-9)^3$

k $(2x+6)^3$

l $(b+3d)^3$

m $(3x+2y)^4$

n $(x+3y)^5$

o $\left(2p + \frac{5}{p}\right)^3$

p $\left(x^2 - \frac{2}{x}\right)^4$

q $\left(q + \frac{2}{p^3}\right)^5$

r $\left(x + \frac{1}{x}\right)^3$

The general term

Recall that: ${}^nC_r = \binom{n}{r}$

We have seen the relationship between Pascal's triangle and its combinatorial equivalent. From this relationship we were able to produce the general expansion for $(x+a)^n$.

That is,

$$(x+a)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}a + \dots + \binom{n}{r}x^{n-r}a^r + \dots + a^n$$

Where the **first** term, $t_1 = x^n$

the **second** term, $t_2 = \binom{n}{1}x^{n-1}a$

the **third** term, $t_3 = \binom{n}{2}x^{n-2}a^2$

:

the **r**th term, $t_r = \binom{n}{r-1}x^{n-(r-1)}a^{r-1}$

The $(r+1)$ th term is also known as the general term.

That is: $t_{r+1} = \binom{n}{r}x^{n-r}a^r$

It is common in examinations for questions to only ask for a part of an expansion. This is because the previous examples are time consuming to complete.

Example A.4.3

Find the 5th term in the expansion $\left(x + \frac{2}{x}\right)^{10}$, when expanded in descending powers of x .

The fifth term is given by t_5 . Using the general term, this means that $r+1 = 5 \Leftrightarrow r = 4$.

For this expansion we have that $n = 10$, therefore,

$$\begin{aligned} t_5 &= \binom{10}{4}(x)^{10-4}\left(\frac{2}{x}\right)^4 = 210 \times x^6 \times \frac{16}{x^4} \\ &= 3360x^2 \end{aligned}$$

Therefore the fifth term is $t_5 = 3360x^2$.

Example A.4.4

Find the term independent of x in the expansion

$$2x + \frac{1}{x^2}$$

In this case we want the term independent of x , that is, the term that involves x^0 .

Again, we first find an expression for the general term,

$$\begin{aligned} t_{r+1} &= \binom{6}{r} (2x)^{6-r} \left(-\frac{1}{x^2}\right)^r \\ &= \binom{6}{r} (2)^{6-r} (x)^{6-r} (-1)^r (x^{-2})^r \\ &= \binom{6}{r} 2^{6-r} (-1)^r x^{6-r-2r} \\ &= \binom{6}{r} 2^{6-r} (-1)^r x^{6-3r} \end{aligned}$$

Notice how we had to separate the constants and the x term.

Next, we equate the power of x in the expansion to 0:
 $6 - 3r = 0 \Leftrightarrow r = 2$.

We therefore want:

$$t_3 = \binom{6}{2} 2^{6-2} (-1)^2 x^{6-6} = 15 \times 16 \times 1 \times x^0 = 240.$$

So, the term independent of x is 240.

Exercise A.4.2

- Find the terms indicated in the expansions of the following expressions.

Expression	Term
a $(x+4)^5$	x^3
b $(x+y)^7$	$x^5 y^2$
c $(2x-1)^8$	x^3
d $(3x-2)^5$	x^4
e $(2-3p^2)^4$	p^4
f $(2p-3q)^7$	$p^2 q^5$
g $\left(3p - \frac{2}{p}\right)^7$	p

- Find the coefficients of the terms indicated in the expansions of the following expressions.

Expression	Term
a $(2x-5)^8$	x^3
b $(5x-2y)^6$	$x^2 y^4$
c $(x+3)^6$	x^3
d $(2p-3q)^5$	$p^4 q$
e $\left(2x - \frac{3}{p}\right)^8$	$\frac{x^2}{p^6}$
f $\left(q + \frac{2}{p^3}\right)^5$	$\frac{q^3}{p^6}$

- Use the first three terms in the expansion of $(1+x)^4$ to find an approximate value for 1.01^4 . Find the percentage error in using this approximation.
- Write the expansion of $(5+2x)^6$.
 - Use the first three terms of the expansion to approximate 5.2^6 .
 - Find the absolute error in this approximation.
 - Find the percentage error in this approximation.
- Find the coefficient of x^{-3} in the expansion of $(x-1)^3 \left(\frac{1}{x} + x\right)^6$
- Find the constant term in the expansion of $\left(x - \frac{1}{2x}\right)^{10}$
- Find the constant term in the expansion of $\left(3x - \frac{1}{6x}\right)^{12}$
- Find the term independent of x in the expansion of $(2-x)^3 \left(\frac{1}{3x} - x\right)^6$
- Find the term independent of x in the expansion of $\left(2x - \frac{1}{x}\right)^6 \left(\frac{1}{2x} + x\right)^6$

10. In the expansion of $\left(x - \frac{a}{x}\right)^5 \left(x + \frac{a}{x}\right)^5$, where a is a non-zero constant, the coefficient of the term in x^{-2} is '-9' times the coefficient in x^2 . Find the value of the constant a .
11. If the coefficient of the x^2 in the expansion of $(1 - 3x)^n$ is 90, find n .
12. Three consecutive coefficients in the expansion of $(1 + x)^n$ are in the ratio 6 : 14 : 21. Find the value of n .
13. Find the independent term in the following expansions:
 a $\left(y + \frac{1}{y}\right)^3 \left(y - \frac{1}{y}\right)^5$ b $\left(2x + 1 - \frac{1}{2x^2}\right)^6$
14. In the expansion of $(1 + ax)^n$ the first term is 1, the second term is $24x$ and the third term is $252x^2$. Find the values of a and n .
15. In the expansion of $(x + a)^3(x - b)^6$, the coefficient of x^7 is -9 and there is no x^8 term. Find a and b .
16. The first three terms in the expansion of $(1 + ax)^n$ are given below. Find the value of a and n .
- a $1 - 18x + 135x^2 \dots$
- b $1 + \frac{5}{3}x + \frac{10}{9}x^2 \dots$
- c $1 + 4\sqrt{2}x + 12x^2 \dots$
- d $1 - \frac{2\sqrt{2}}{3}x + \frac{1}{3}x^2 \dots$

Answers



This chapter has looked at some patterns of numbers.

Such patterns are important in practical terms, business finance etc.

One of the most commonly encountered sets of numbers is the prime numbers.

The prime numbers are fundamental to mathematics. A prime number has exactly two factors, itself and 1. Thus 1 is not a prime number.

With this definition, we can move on to the statement of the **Fundamental Theorem of Arithmetic**:

Every whole number can be expressed uniquely as a product of prime numbers.

Thus $15 = 3 \times 5$ where both 3 and 5 are prime numbers and there is no other product of primes that will produce 15.

Likewise, $24 = 2 \times 2 \times 2 \times 3$ and there is no other set of primes that will do this.

If we allowed 1 to be prime, this important theorem would no longer be true as $15 = 1 \times 3 \times 5$ would be a second prime version of 15.

Prime numbers are also used in the encryption system used to secure cash card transactions. These depend on numbers like 15 that have exactly two prime factors. If these numbers are huge, it can be virtually impossible to find the two factors. Knowing the factors makes it easy to find their product.

To put it another way, given 3 and 5, it is easy to find 15. Given 15, it is much harder to find a factor. This becomes extremely difficult if the numbers concerned have dozens of digits. We have a reversible process that is easy in one direction and very hard in the reverse direction, unless one has the key.

Aside from these practical uses of prime numbers, they provide a genuine puzzle. Like all good puzzles, their 'rules' are not too difficult to understand. However, a bit like chess, mastering the 'game' is proving profoundly difficult.

Very little is known about the distribution of the prime numbers. This is despite the problem having received

attention from many of the world's greatest mathematicians. We have, for example, no test for whether or not a number is prime - other than systematically testing factors. There is no exact formula for the number of primes less than any given number. There are some approximations, but that is all.

We do know that there are an infinite number of prime numbers, a result proved by Euclid over 2,000 years ago.

Suppose that there are a finite number of primes: 2, 3, 5, 7, ..., P .

The product $2 \times 3 \times 5 \times \dots \times P$ is divisible by every prime number. It follows that the number $2 \times 3 \times 5 \times \dots \times P + 1$ is divisible by none of the primes and therefore must be a prime that we had missed from our list. The list must have been incomplete. This argument can then be repeated indefinitely, proving that there must be an infinite number of primes.

Aside from the mystery surrounding the distribution of primes, there are other unsolved problems.

For example, some pairs of prime numbers are separated by a single (non-prime) even number. These pairs are known as **twin primes**. 5 & 7 and 11 & 13 are examples of twin primes. Are there an infinite number of twin primes? This seems to be

a simple question, but nobody knows the answer!

At the time of writing, the largest known prime number is $2^{74,207,281} - 1$, a number with 22,338,618 digits (large enough to fill this book around 20 times over). As we have discussed, there are an infinite number of primes bigger than this.

The absence of a pattern in the primes is a mystery. One of the earliest ways of listing the primes is known as the **Sieve of Eratosthenes**.

Begin by setting out all the whole numbers in a grid. Then cross out all the multiples of 2 except 2 (they appear in columns). Next cross out all the multiples of 3 except 3 (which appear on diagonals). It is not necessary to do this for 4 (why?). Repeat for 5 and all the other prime numbers until there are no more crossings out. The prime numbers remain - they have been 'sieved' out.

We have done the multiples of 2 (orange) and the multiples of 3 (green) below. Some numbers are crossed out twice (blue). However, the question that this raises is...

The whole numbers are arranged in a pattern. They are crossed out systematically. Why is there no evident pattern in the un-crossed out numbers?

0	1	2	3	4	5	6	7	8	
10	11	12	13	14		16	17	18	19
20		22	23	24	25	26		28	29
30	31	32		34	35	36	37	38	
40	41	42	43	44		46	47	48	49
50		52	53	54	55	56		58	59
60	61	62		64	65	66	67	68	
70	71	72	73	74		76	77	78	79
80		82	83	84	85	86		88	89
90	91	92		94	95	96	97	98	

A.8 Proof

SL 1.6

Axioms

Mathematics is based on axioms. These are 'facts' that are assumed to be true. An axiom is a statement that is accepted without proof. Early sets of axioms contained statements that appeared to be obviously true. Euclid postulated a number of these 'obvious' axioms.

An example of an axiom is:

'Things equal to the same thing are equal to each other'; That is, if $y = a$ and $x = a$ then $y = x$.

Euclid was mainly interested in geometry and we still call plane geometry 'Euclidean'. In Euclidean space, the shortest distance between two points is a straight line. We will see later that it is possible to develop a useful, consistent mathematics that does not accept this axiom.

Most axiom systems have been based on the notion of a 'set', meaning a collection of objects. An example of a set axiom is the 'axiom of specification'. In crude terms, this says that if we have a set of objects and are looking at placing some condition or specification on this set, then the set thus specified must exist. An example of this axiom is:

Assume that the set of citizens of China is defined. If we impose the condition that the members of this set must be female, then this new set (of Chinese females) is defined.

As a more mathematical example, if we assume that the set of whole numbers exists, then the set of even numbers (multiples of 2) must also exist.

Mathematics has, in some sense, been a search for the smallest possible set of consistent axioms. It is an unusual pursuit in this respect. Pure mathematics is concerned with absolute

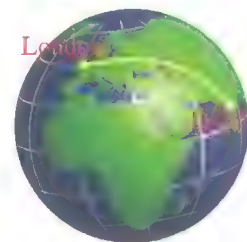
truth only in the sense of creating a self-consistent structure of thinking.

As an example of some axioms that may not seem to be sensible, consider a geometry in which the shortest path between two points is the arc of a circle and all parallel lines meet. These 'axioms' do not seem to make sense in 'normal' geometry. The first mathematicians to investigate non-Euclidean geometry were the Russian, Nicolai Lobachevsky (1792–1856) and the Hungarian, Janos Bolyai (1802–60).

Independently, they developed self-consistent geometries that did not include the so called parallel postulate which states that for every line AB and point C outside AB there is only one line through C that does not meet AB.



Since both lines extend to infinity in both directions, this seems to be 'obvious'. Non-Euclidean geometries do not include this postulate and assume either that there are no lines through C that do not meet AB or that there is more than one such line. It was the great achievement of Lobachevsky and Bolyai that they proved that these assumptions lead to geometries that are self consistent and thus acceptable as 'true' to pure mathematicians. In case you are thinking that this sort of activity is completely useless, one of the two non-Euclidean geometries discussed above has actually proved to be useful; the geometry of shapes drawn on a sphere. This is useful because it is the geometry used by the navigators of aeroplanes and ships.



Proof

Proof has a very special meaning in mathematics. We use the word generally to mean 'proof beyond reasonable doubt' in situations such as law courts when we accept some doubt in a verdict. For mathematicians, proof is an argument that has no doubt at all. When a new proof is published, it is scrutinized and criticized by other mathematicians and is accepted when it is established that every step in the argument is legitimate. Only when this has happened does a proof become accepted.

Technically, every step in a proof rests on the axioms of the mathematics that is being used. As we have seen, there is more than one set of axioms that could be chosen. The statements that we prove from the axioms are known as theorems. Once we have a theorem, it becomes a statement that we accept as true and which can be used in the proof of other theorems. In this way we build up a structure that constitutes a 'mathematics'. The axioms are the foundations and the theorems are the superstructure. In the previous section we made use of the idea of consistency.

This means that it must not be possible to use our axiom set to prove two theorems that are contradictory.

There are a variety of methods of proof. This section will look at some of these in detail. We will mention others.

Rules of Inference

All proofs depend on rules of inference. Fundamental to these rules is the idea of 'implication'.

As an example, we can say that $2x = 4$ (which is known as a proposition) implies that $x = 2$ (provided that x is a normal real number and that we are talking about normal arithmetic). In mathematical shorthand we would write this statement as $2x = 4 \Rightarrow x = 2$.

This implication works both ways because $x = 2$ implies that $2x = 4$ also. This is written as $x = 2 \Rightarrow 2x = 4$, or the fact that the implication is both ways can be written as $x = 2 \Leftrightarrow 2x = 4$. The symbol $\Leftrightarrow$ is read as 'If and only if' or simply as 'Iff', i.e. If with two fs.

There are four main rules of inference:

1. **The rule of detachment:** from a is true and $a \Rightarrow b$ is true we can infer that b is true where a and b are propositions.

Example

If the following propositions are true:

It is raining.

If it is raining, I will take an umbrella.

We can infer that I will take an umbrella.

2. **The rule of syllogism:** from $a \Rightarrow b$ is true and $b \Rightarrow c$ is true, we can conclude that $a \Rightarrow c$ is true. a , b and c are propositions.

Example:

If we accept as true that:

if x is an odd number then x is not divisible by 4
($a \Rightarrow b$) and,

if x is not divisible by 4 then x is not divisible by 16
($b \Rightarrow c$)

We can infer that the proposition:

if x is an odd number then x is not divisible by 16
($a \Rightarrow c$) is true.

3. **The rule of equivalence:** at any stage in an argument we can replace any statement by an equivalent statement.

Example:

If x is a whole number, the statement x is even could be replaced by the statement x is divisible by 2.

4. **The rule of substitution:** If we have a true statement about all the elements of a set, then that statement is true about any individual member of the set.

Example:

If we accept that all lions have sharp teeth then Benji, who is a lion, must have sharp teeth.

Now that we have our rules of inference, we can look at some of the most commonly used methods of proof

Proof by Exhaustion

This method can be, as its name implies, exhausting! It depends on testing every possible case of a theorem.

Example

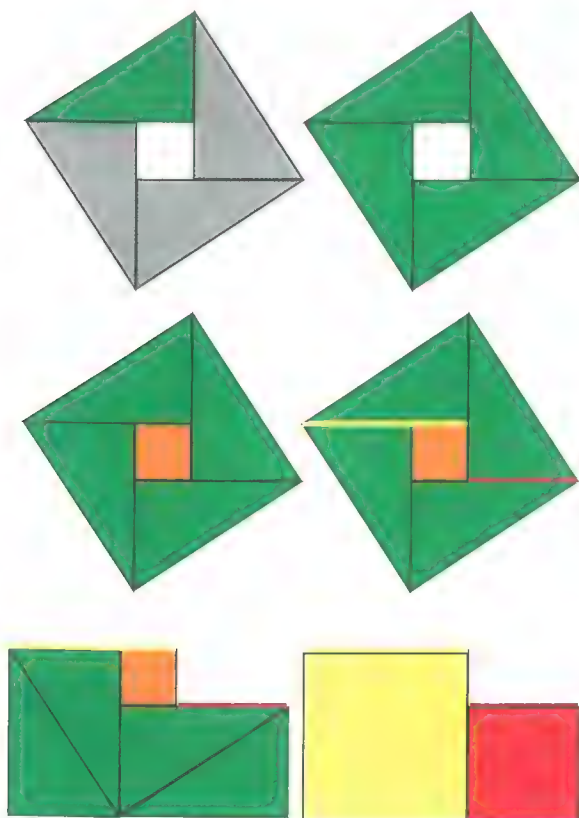
Consider the theorem: Every year must contain at least one 'Friday the thirteenth'.

There are a limited number of possibilities as the first day of every year must be a Monday or a Tuesday or a Wednesday ... or a Sunday (seven possibilities). Taking the fact that the year may or may not be a leap year (with 366 days) means that there are going to be fourteen possibilities.

Once we have established all the possibilities, we would look at the calendar associated with each and establish whether or not it has a 'Friday the thirteenth'. If, for example, we are looking at a non-leap year in which January 1st is a Saturday, there will be a 'Friday the thirteenth' in May. Take a look at all the possibilities (an electronic organizer helps!). Is the theorem true?

Direct Proof

The diagrams below represent a proof of the theorem of Pythagoras described in *The Ascent of Man* (Bronowski, pp. 158–61). The theorem states that the area of a square drawn on the hypotenuse of a right-angled triangle is equal to the sum of the areas of the squares drawn on the two shorter sides. The method is direct in the sense that it makes no assumptions at the start. Can you follow the steps of this proof and draw the appropriate conclusion?



Notation

Written Mathematics tends to use a lot of 'notation' or 'shorthand'.

Here is a short list of symbols frequently used in proofs:

Equality

The symbol $=$ means that two quantities are equal. It was invented by a Welshman, Robert Recorde (1512–1558).

Thus $2x + 3 = 7$ is a statement of equality.

One important feature of equalities is that they are not necessarily always true. Our statement is true if $x = 2$ but false if $x = 1$.

Identity

The statement $2(x + 3) \equiv 2x + 6$ is an identity. It is true for all values of x . The triple barred equals sign emphasises that it is a stronger statement of equality than $=$.

'Not equal to' is written $\neq$. So $5 \neq 7$.

Implication

There is a commonly used notation that is used when we want to say that one true statement implies the truth of another.

It is raining therefore I will wear a raincoat is shortened to:

'It is raining' $\Rightarrow$ 'I will wear a raincoat'. The symbol $\Rightarrow$ is read 'implies'.

Note that an implication does not necessarily work in reverse. If I wear a raincoat, it is not necessarily raining.

If an implication is two way, we use a double headed arrow.

If $x = 2$ then $2x = 4$ works both ways. If $2x = 4$ then $x = 2$. This is shortened to $x = 2 \Leftrightarrow 2x = 4$

Example A.8.1

Classify the relationship between these pairs of expressions as $\neq$, $=$ or $\equiv$.

- | | | | |
|---|--------------------------|---|------------------------------|
| a | $\sqrt{x^2 + 1}, x + 1$ | b | $(x + 1)^2, x^2 + 2x + 1$ |
| c | $\frac{1}{3}, 0.\dot{3}$ | d | $\sqrt{x^2 + 2x + 1}, x + 1$ |

a $\sqrt{x^2+1}, x+1$

It is common to think that these two expressions are either equal or identical. If they are, they must be equal whatever value of x we choose. Suppose we choose 2.

$$\sqrt{2^2+1} = \sqrt{5} \text{ and } 2+1=3$$

$$\sqrt{x^2+1} \neq x+1$$

b $(x+1)^2, x^2+2x+1$

This is true for all x . $(x+1)^2 \equiv x^2+2x+1$

c $\frac{1}{3}, 0.\dot{3}$

These two quantities are identically equal $\frac{1}{3} \equiv 0.\dot{3}$.

d $\sqrt{x^2+2x+1}, x+1$

This is a more complex example. We need to be careful about the fact that 'square root' returns two answers. If we allow that, then $\neq$ is the correct answer.

Suppose we only take the positive value of the square root. This will be true for $x \geq 0$. What about negative values? Taking particular values does not provide a general proof. It can, however get things started.

$$\sqrt{(-2)^2+2 \times (-2)+1} = \sqrt{4-4+1} = 1 \text{ and}$$

$$-2+1=-1.$$

Example A.8.2

Place either $\Rightarrow$ or $\Leftrightarrow$ between these pairs of statements as appropriate.

a $x \in \mathbb{N}$ x is even, $x \in \mathbb{N}$ x is divisible by 2.

b $a > 7, a > 5$.

c $p > 2$ is prime, p is odd.

d $x^2 = 49, x = 7$.

a x is specified as being a Natural Number, 1, 2, 3, 4, ... so we do not need to worry about whether or not -2 is even. This is an implication that runs both ways:

$$x \in \mathbb{N} \text{ } x \text{ is even} \Leftrightarrow x \in \mathbb{N} \text{ } x \text{ is divisible by 2.}$$

b Numbers bigger than 7 are also bigger than 5 but not the other way round.

$$a > 7 \Rightarrow a > 5$$

c 2 is the only even prime and is included. This is a one-way implication.

$$p > 2 \text{ is prime} \Rightarrow p \text{ is odd}$$

d This only works one way (because of the negative option for square root).

$$x = 7 \Rightarrow x^2 = 49$$

Exercise A.8.1

1. Classify these statements as true/false:

a x is odd $\Rightarrow x$ is divisible by 3.

b x is odd $\Rightarrow x^2$ is odd.

c x is even $\Rightarrow x^2$ is even.

d x is prime $\Leftrightarrow x^2$ has exactly two factors.

e $\frac{1}{x+1} - \frac{2}{x} \equiv \frac{-(x+2)}{x(x+1)}, x > 0$

f x is divisible by 2 & 3 $\Rightarrow x$ is divisible by 6.

g $(x+1)^3 - (x-1)^3 \equiv 2$.

2. Which of these statements of equality are also identities?

a $2x + 3 = 7$

b $x(x-2) = x^2 - 2x$

c $x = \sqrt{x^2}$

d $\frac{1}{x-1} - \frac{2}{x+1} \equiv \frac{-(x-3)}{x^2-1}, x > 0$

e $(a^x)^y = a^{xy}$

f $\sqrt[a]{b} = b^a$

g $\frac{1}{a} - \frac{1}{b} + \frac{1}{c} = \frac{a-b+c}{abc}$

Geometric Proofs

One of the best ways of practising the construction of direct proofs is to look at geometric examples. These depend on the axioms of Euclid and a number of geometric theorems.

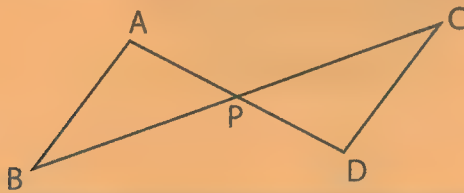
Whilst they are not strictly on the syllabus, these proofs also played an important part in the long history of Mathematics. The following problems use Mathematics that you should have encountered during your Middle School Years.

A summary of some important facts can be found here:



Example A.8.3

Given that $AP = DP$ and $BP = CP$, prove that the triangles ABP and CDP are congruent.



Proofs should proceed from accepted truths through a chain of implication to the desired statement.

In this case, 'accepted truth' includes the data in the question as well as previously proved theorems such as Pythagoras.

The data in this case $AP = DP$ and $BP = CP$.

$\angle APB = \angle BPC$ (Vertically Opposite Angles)

$\Rightarrow \triangle ABP$ and $\triangle CDP$ are congruent (Side-Angle-Side criterion).

Notice that both of the implication have justifications. This is a key feature of all mathematical proofs - no statements are accepted as true unless they can be justified.

We were not asked to do this but some further inferences can be drawn:

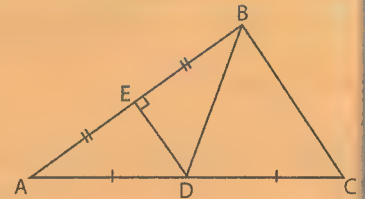
Since the triangles are congruent, it follows that $\angle ABP = \angle DCP$

Note that we must be careful to pair these angles correctly. It is this pair that are equal because they are opposite AP and DP and $AP = DP$.

$\angle ABP = \angle DCP \Rightarrow AB$ is parallel to CD (alternate angles).

Example A.8.4

Prove that $\triangle DBC$ is isosceles



In $\triangle AED$ and $\triangle BED$:

$AE = BE$ (given)

ED is common to both triangles

$\angle AED = \angle BED = 90^\circ$ (given) $\Rightarrow \triangle AED$ and $\triangle BDE$ are congruent (side-angle-side criterion)

$AD = BD$

$AD = DC$ (given) $\Rightarrow$

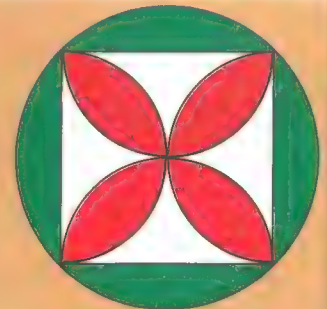
$BD = DC$ (rule of syllogism) $\Rightarrow$

$\triangle BDC$ is isosceles

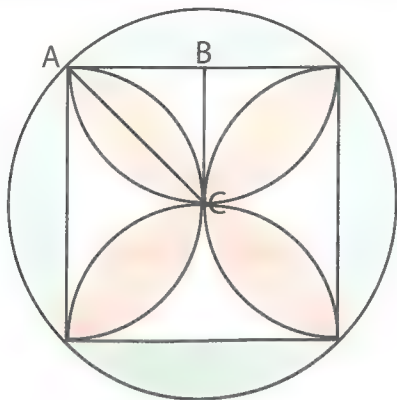
Example A.8.5

A square is inscribed in a circle. The red areas are bounded by semicircles.

Prove that the red shaded area is equal in magnitude to the green shaded area.



Be careful of falling for superficial proofs such as "There are four red bits and four green bits so the areas are the same".



Let the square be 2 by 2.

The area of the square is $2 \times 2 = 4$ square units (u^2).

$AB = BC = 1$ by symmetry.

$AC^2 = AB^2 + BC^2$ (Pythagoras)

$AC^2 = 1^2 + 1^2$

$\Rightarrow AC = \sqrt{2}$

Area of the large circle $= \pi r^2$

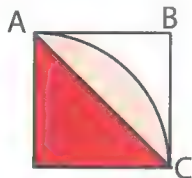
$$= \pi \times (\sqrt{2})^2$$

$$= 2\pi (u^2)$$

Green shaded area $=$ area of circle $-$ area of square

$$2\pi - 4 (u^2)$$

To find the area of the red 'petal shapes'.



The area with the two types of red shading is a quarter circle of radius 1.

Red shaded area $= \frac{1}{4}\pi r^2$

$$= \frac{1}{4}\pi \times 1^2$$

$$= \frac{1}{4}\pi$$

The surrounding square is 1 by 1 and has area 1 (u^2)

The white area bounded above $= 1 - \frac{1}{4}\pi$

Returning to the first diagram the red shaded area is the big square (2 by 2) minus 8 of these white areas.

The red area $= 4 - 8 \times (1 - \frac{1}{4}\pi)$

$$= -4 + 2\pi$$

$$= 2\pi - 4.$$

It follows that the red and green areas are the same.

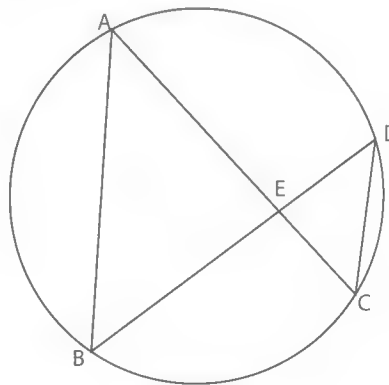
If you feel that the step we made right at the start (assuming a size for the square) was not legitimate, you are probably right.

This can all be put right by letting the size of the square be x .

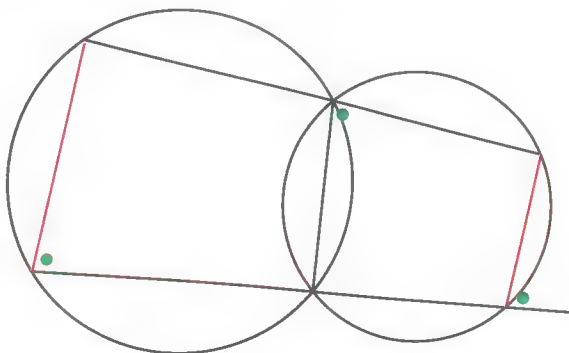
The whole proof can then proceed as before. The technique of tackling a difficult proof by simplifying it first, developing a proof and then using the same strategy to do the general case can be very useful.

Exercise A.8.2

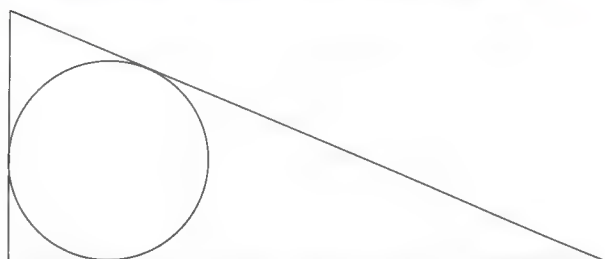
1. Prove that if $\angle BAC = \angle ACD$, triangles ABE and CDE are similar.



2. Prove that if the angles marked with the green dots are equal, then the red lines are parallel.

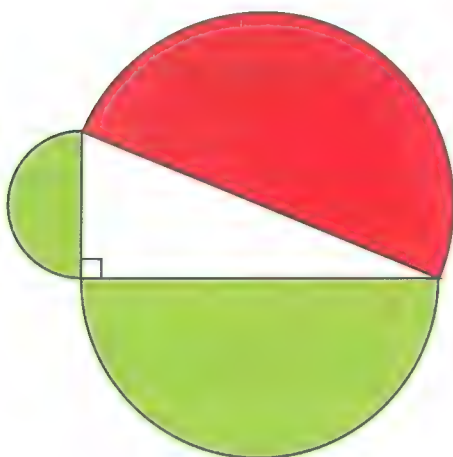


3. A circle is inscribed in a 5, 12, 13 right triangle



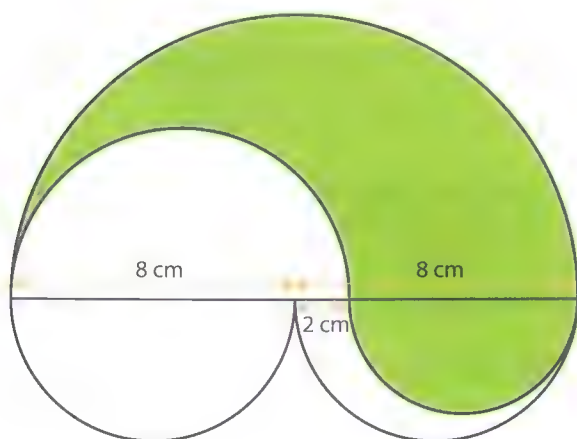
Prove that the radius of the circle = 2.

4. Three semicircles are drawn on the sides of a right angled triangle. Prove that the sum of the areas of the two smaller circles (green) is equal to the area of the large semicircle (red).

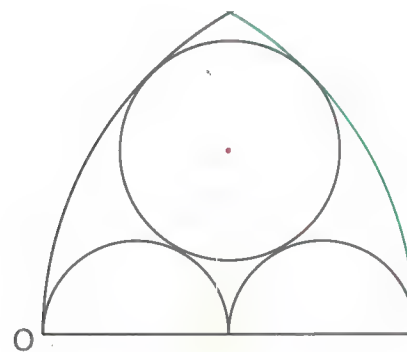


5. The figure consists of five semicircles all centred on the blue line. Prove that the white and green areas have:

- Equal areas
- Equal perimeters.



6. Gothic Tracery. The diagram shows a pattern common in gothic window designs.



The main arch consists of the arcs of two circles of radius a . The green arc is centred at O .

Prove that $b = \frac{a\sqrt{6}}{5}$.

7. A right triangle has area A and perimeter $2P$. Prove that the hypotenuse is given by:

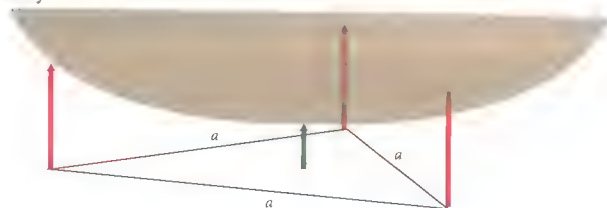
$$P - \frac{A}{P}$$

8. Two solids are made from twelve congruent equilateral triangles.

The first is a tetrahedron of volume T and the second is an octahedron of volume O .

Prove that $O = 4T$.

9. A spherometer is a device for measuring the radius of curvature of objects such as lenses. It consists of a triangular device with three prongs arranged in an equilateral triangle. A fourth prong can be screwed up and down so that all four make contact with the object.



The green prong makes contact with the lens (which has radius r) when it is h below the plane of the red prongs. Prove that:

$$r = \frac{a^2}{6h} + \frac{h}{2}$$

Numerical Proofs

Proofs of theorems involving numbers almost always deal with infinite sets. This means that 'proof by exhaustion' is seldom an option.

It is also the case that there are some superficially simple statements that have been very resistant to proof. Some examples are:

Twin Primes

Pairs of prime numbers such as 11 & 13 that are separated by one even number are said to be 'twin primes'. The truth of the statement 'There are an infinite number of twin primes' is unresolved.

Goldbach's Conjecture

Every even number greater than 2 can be written as the sum of two prime numbers. For example: $6=3+3$, $8=3+5$ etc.

This is unproved.

Fermat's Last Theorem

$a^n + b^n = c^n$, $a, b, c, n \in \mathbb{N}$ has no solutions for $n > 2$.

This has been proved recently. It was proposed in 1637 so it took over 300 years of trying!

The approaches we have been using (working from the data through a string of inferences to the required statement) work for these problems too.

Example A.8.6

Prove that for any four consecutive integers a, b, c, d :

$ab + ac + ad + bc + bd + cd + 1$ is divisible by 12.

Let:

$$a = n-1, b = n, c = n+1, d = n+2$$

$$ab + ac + ad + bc + bd + cd + 1$$

$$= (n-1)n + (n-1)(n+1) + (n-1)(n+2) + n(n+1)$$

$$+ \dots + n(n+2) + (n+1)(n+2) + 1$$

$$= n^2 - n + n^2 - 1 + n^2 + n - 2 + n^2 + n + n^2 + 2n + n^2 + 3n + 2 + 1$$

$$= 6n^2 + 6n$$

$$= 6n(n+1)$$

$n(n+1)$ is the product of an odd and even number and so is even.

$6 \times$ an even number is divisible by 12, so:

$$ab + ac + ad + bc + bd + cd + 1 \text{ is divisible by 12.}$$

Example A.8.7

The test for divisibility by 9 is:

Add the digits to get a digit sum. If the digit sum is divisible by 9, then the original number is divisible by 9.

We have suggested that a good way of approaching a general proof of this sort is to look at a particular example, frame a proof for that and use the strategy to complete the full truth.

Let us look at the problem of the divisibility by 9 of 567 (which has a digit sum of $5 + 6 + 2 = 18$).

According to the test, since 18 is divisible by 9, then 567 is divisible by 9.

In constructing the proof, write 567 as:

$$567 = 5 \times 10^2 + 6 \times 10^1 + 7 \times 10^0$$

Consider what happens if we divide every term in this equation by 9. What we are interested in is what the remainders will be.

Since 9, 99, 999, 9 999 etc. are all divisible by 9, it follows that $10^0, 10^1, 10^2, 10^3$ etc. leave a remainder of 1 on division by 9.

Likewise, 5 leaves a remainder of 5 on division by 9, 6 leaves a remainder of 6 on division by 9 etc.

So the remainders of the right hand side are:

$$5 \times 1 + 6 \times 1 + 7 \times 1 = 5 + 6 + 7 = 18.$$

If the remainder is divisible by 9, then there is no remainder at all.

Generalising:

$$\text{If } x = a_0 \times 10^0 + a_1 \times 10^1 + a_2 \times 10^2 + a_3 \times 10^3 + \dots$$

The remainders on division by 9 are:

$$a_0 \times 1 + a_1 \times 1 + a_2 \times 1 + a_3 \times 1 + \dots \text{ (ie. the digit sum).}$$

So, x is divisible by 9 if and only if the digit sum is divisible by 9.

Note that this is a theorem that can be successively applied.

383, 942 has a digit sum of 29 which has a digit sum of 11 which has a digit sum of 2 which is not divisible by 9. Hence,

383, 942 is not divisible by 9.

Example A.8.8

Prove that between any two rational numbers there is at least one rational number.

What further can you conclude?

Again, we look at a specific example.

A simplified version is:

Find a rational number that is between $\frac{2}{3}$ and $\frac{3}{4}$.

The mean of two numbers is always between them.

$$\begin{aligned} \text{The mean of } \frac{2}{3} \text{ and } \frac{3}{4} \text{ is } \frac{\frac{2}{3} + \frac{3}{4}}{2} &= \frac{\frac{2 \times 4}{3 \times 4} + \frac{3 \times 3}{4 \times 3}}{2} \\ &= \frac{\frac{2 \times 4 + 3 \times 3}{2 \times 3 \times 4}}{2} \\ &= \frac{17}{24}. \end{aligned}$$

If we want to check that this fraction is between $\frac{2}{3}$ and $\frac{3}{4}$, we need to use LCMs.

$$\begin{aligned} \frac{2}{3} &= \frac{16}{24}, \quad \frac{17}{24} < \frac{18}{24} \\ \frac{3}{4} &= \frac{18}{24} \end{aligned}$$

It is evident that: $\frac{16}{24} < \frac{17}{24} < \frac{18}{24}$

Thus, we have found a rational number between $\frac{2}{3}$ and $\frac{3}{4}$.

Now, follow this pattern to prove the general case:

Let the two rational numbers be: $\frac{a}{b}, \frac{c}{d}, a, b, c, d \in \mathbb{Z}$.

with $\frac{a}{b} < \frac{c}{d}$.

$$\begin{aligned} \text{The mean of these two numbers is: } \frac{\frac{a}{b} + \frac{c}{d}}{2} &= \frac{\frac{ad}{bd} + \frac{bc}{bd}}{2} \\ &= \frac{ad + bc}{2bd} \end{aligned}$$

Since a, b, c, d are all integers, both the numerator and denominator of this expression are integers.

Thus $\frac{ad + bc}{2bd}$ is a rational number between $\frac{a}{b}, \frac{c}{d}$.

If you accept that a mean of two numbers must lie between them, the proof ends here.

Note that we can conclude that there must be another rational number between the lower number and the mean. There is also yet another rational between the mean and the larger number.

Successive use of this theorem means that we can infer that, between any two rational numbers, there are an infinite number of other rational numbers.

The check (which is not really necessary) is a bit more complex as we may be dealing with cases in which some of a, b, c, d are negative.

$$\begin{aligned} \text{If } a, b, c, d > 0: \quad \frac{a}{b} &< \frac{ad + bc}{2bd} < \frac{c}{d} \\ \frac{2ad}{2bd} &< \frac{ad + bc}{2bd} < \frac{2bc}{2bd} \\ 2ad &< ad + bc < 2bc \end{aligned}$$

This contains two propositions: $2ad < ad + bc$
and $ad + bc < 2bc$

$$\begin{aligned} ad &< bc \end{aligned}$$

That is, there is a single proposition here: $ad < bc$.

We have to be careful multiplying both sides of the inequality by $2bd$ in case this is negative. We have already specified that it is not.

Since we started with: $\frac{a}{b} < \frac{c}{d}$, it follows that: $\frac{ad}{bd} < \frac{bc}{bd}$.

Since bd is positive: $ad < bc$, which is what we need.

If bd is negative, inequality signs must be reversed if there is multiplication by it.

$$\frac{2ad}{2bd} < \frac{ad+bc}{2bd} < \frac{2bc}{2bd} \text{ becomes } 2ad > ad+bc > 2bc.$$

As before, this resolves to the single statement: $ad > bc$.

But the original premise also leads to this:

$$\frac{a}{b} < \frac{c}{d}$$

$$\frac{ad}{bd} < \frac{bc}{bd}$$

$$ad > bc$$

and the proof is complete.

Exercise A.8.3

1. Prove that every number that ends in 0 or 5 is divisible by 5.
2. Prove that every square number can be written as the sum of two triangle numbers.
3. Does this pattern continue?

$$1^3 = 1^2$$

$$1^3 + 2^3 = (1 + 2)^2$$

$$1^3 + 2^3 + 3^3 = (1 + 2 + 3)^2$$

$$1^3 + 2^3 + 3^3 + 4^3 = (1 + 2 + 3 + 4)^2$$

4. Generalise and prove that this pattern continues:

$$1 + 1 + 1 = 3$$

$$1 + 2 + 3 + 2 + 1 = 9$$

$$1 + 3 + 6 + 7 + 6 + 3 + 1 = 27$$

5. Generalise and prove that this pattern continues:

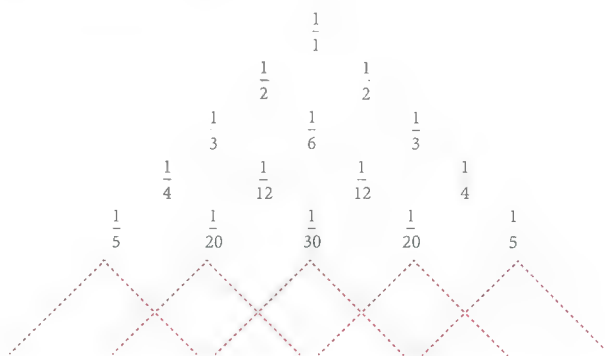
$$1 - 1 + 1 = 1$$

$$1 - 2 + 3 - 2 + 1 = 1$$

$$1 - 3 + 6 - 7 + 6 - 3 + 1 = 1$$

6. Prove that x is divisible by 11 if and only if the alternating sum of its digits $a_0 - a_1 + a_2 - a_3 + a_4 + \dots + a_m(-1)^m$ is divisible by 11.

7. Prove that the sum of the digits of the digits of the sum of the digits of the sum of the digits of 4444⁴⁴⁴⁴ is 7.
8. Prove that there exist rational numbers A & B such that A^B is irrational.
9. Following on from the previous question, prove that there exist irrational numbers A & B such that A^B is rational.
10. If one million factorial is written in full, how many zeros does it end in?
11. If a , b and c are integers such that $a|b$ and $b|c$, then $a|c$ means a divides into c .
12. Prove that if m is an integer then 3 divides $m^3 - m$.
13. Two integers are said to be relatively prime if their greatest common divisor is 1.
 - i Show that if m is a positive integer, then $3m + 2$ and $5m + 3$ are relatively prime.
 - ii Show that if a and b are relatively prime integers, then the greatest common divisor of $a + 2b$ and $2a + b = 1$ or 3.
14. The diagram shows the pattern known as Leibniz's Harmonic Triangle. It is cousin to Pascal's Triangle. Can you see how it 'works'?



Use the triangle to investigate the truth, or otherwise, of these postulates.

- i $\frac{1}{1} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \dots$
- ii $\frac{1}{2} = \frac{1}{3} + \frac{1}{12} + \frac{1}{30} + \frac{1}{60} + \frac{1}{105} + \dots$

$$\text{iii} \quad \frac{1}{3} = \frac{1}{4} + \frac{1}{20} + \frac{1}{60} + \frac{1}{140} + \frac{1}{280} + \dots$$

Algebraic Proofs

We have already been using algebra to construct proofs as they relate to postulates involving numbers and geometric shapes.

Here are two further examples:

Example A.8.9

Prove that for $n \in \mathbb{N}$, $(3n+1)^2 - (3n-1)^2$ is a multiple of 4.

$$\begin{aligned} (3n+1)^2 - (3n-1)^2 &= 9n^2 + 6n + 1 - (9n^2 - 6n + 1) \\ &= 12n \end{aligned}$$

n is a whole number, 12 is a multiple of 4 so $12n$ is also a multiple of 4.

Example A.8.10

Prove that the sum of the squares of any two consecutive even numbers is divisible by 4.

For a natural number, $2n$ is even. The next even number is $2n+2$.

$$\begin{aligned} \text{The sum of the squares is: } (2n)^2 + (2n+2)^2 &= 4n^2 + 4n^2 + 4n + 4 \\ &= 4(2n^2 + n + 1) \end{aligned}$$

Since 4 is divisible by 4 and the bracket is a whole number, the expression is divisible by 4 as required.

Exercise A.8.4

1. Show that the sum of any three consecutive even numbers is a multiple of 6.
2. Show that $n \in \mathbb{N}$, $(2n+3)^2 - (2n-3)^2$ is divisible by 8.

3. Compare the two series:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots$$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \dots$$

Hence prove that: $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$

4. Prove that every recurring decimal can be written as a mixed number and is, hence, rational. Hence conclude that the decimal representation of π is not a recurring decimal.

5. Prove that the conjecture:

For all natural numbers n , $n^n > n!$ is false.

6. Investigate the truth, or otherwise, of Stirling's Formula:

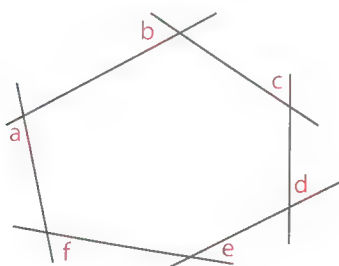
$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

7. Prove that $\frac{(k+2)!}{(k-1)!} = k^3 + 3k^2 + 2k, k > 1, k \in \mathbb{N}$.
8. Prove that a set with n members has 2^n subsets (including the set itself and the empty set).
9. Use the laws of indices to prove that for non-zero a , a^0 is 1.
10. Consider $\lim_{n \rightarrow \infty} (\sqrt{n(n+1)} - n)$.

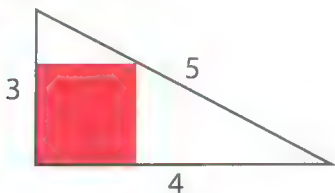
Hence prove that π can be the limit of a sequence of numbers of the form $\sqrt{n} - \sqrt{m}$.

Exercise A.8.5

1. Prove that $a+b+c+d+e+f=360^\circ$.



2. A square is inscribed in a 3, 4, 5 triangle. Prove that its side length is $1\frac{5}{7}$ units.

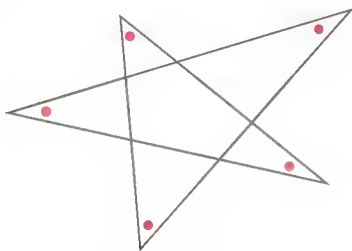


3. A '7 smooth number' is one that factorises into prime numbers less than or equal to 7.

Prove that 86 436 000 is 7 smooth.

4. Prove that $2^n + 1$ where n is an odd number is divisible by 3.

5. Prove that the sum of the angles marked with the red dots is 180° .



6. Prove that there are at least two positive integers $a, b > 5$ such that $a^3 + b^3 = a^4$.

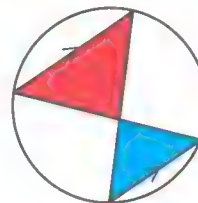
7. Prove that if n is an odd positive integer $n^4 - 18n^2 + 17$ is divisible by 64.

8. The binary operation $\circ$ is defined on the set $\{a, b, c, d\}$ by this table:

$\circ$	a	b	c	d
a	a	c	d	b
b	c	b	a	d
c	d	a	c	a
d	b	d	a	d

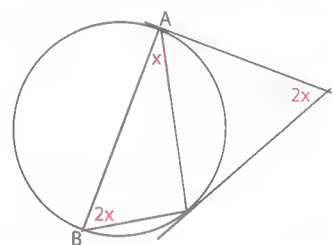
Prove that the operation is commutative i.e. $X \circ Y = Y \circ X$ for all members of the set.

9. Prove that the two coloured triangles are similar.



10. Prove that if n is not divisible by 7, then either $n^3 + 1$ or $n^3 - 1$ is divisible by 7.

11. Prove that AB is a diameter of the circle.



12. Two circles have radii 1 and $3+2\sqrt{2}$ and touch externally. Prove that their common tangents are perpendicular.

13. If $p_1, p_2, p_3, \dots, p_n$ are the first n prime numbers, prove that: $p_1 \times p_2 \times p_3 \times \dots \times p_n + 1$ is also prime and hence prove that there are an infinite number of prime numbers.

14. $a > 0, b > 0, c > 0$ and $a+b+c=2$ prove $abc \leq 1$.

15. Prove that an integer that ends with 7 cannot be a perfect square.

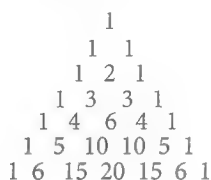
16. If a and b are integers and b is odd, prove that $x^2 + 2ax + 2b = 0$ has no rational roots.

17. Prove that for $n \in \mathbb{Z}^+, n! < \left(\frac{n+1}{2}\right)^n$.



Pascal and Fibonacci

Pascal's Triangle is the pattern shown here. It is of considerable importance in Probability Theory, The Binomial Theorem etc.



The numbers down the side are all 1. The numbers in the body of the triangle are such that each number (eg. yellow) is the sum of the two numbers above and to either side.



If we compare the combinatorial numbers (which we encountered when studying the Binomial Theorem) with the numbers in the Pascal Triangle, there is a superficial similarity.

$$\begin{array}{ccccccc} & & {}^0C_0=1 & & & & \\ & & {}^1C_0=1 & & {}^1C_1=1 & & \\ & & {}^2C_0=1 & & {}^2C_1=2 & & {}^2C_2=1 \\ & & {}^3C_0=1 & & {}^3C_1=3 & & {}^3C_2=3 & & {}^3C_3=1 \\ & & {}^4C_0=1 & & {}^4C_1=4 & & {}^4C_2=6 & & {}^4C_3=4 & & {}^4C_4=1 \end{array}$$

There are a number of things that we need to prove if we want to establish that the resemblance is more than superficial.

The first is that the beginning and end numbers in each row are 1.

This means that, if we are to prove the whole thing, we start by proving:

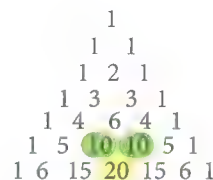
$${}^nC_0 = {}^nC_n = 1, n \in \mathbb{N}$$

Hint: ${}^nC_r = \frac{n!}{(n-r)!r!}$.

The hard bit is the pattern in the body of the table.

We have advised you to tackle difficult proofs like this by taking a particular case, proving that and then using the same strategy to prove the general case.

Taking the example marked in yellow and green in the diagram:



The numbers in green are: ${}^5C_2=10$ and ${}^5C_3=10$.

The number in yellow is: ${}^6C_3 = 20$.

Just using a calculator and observing that it 'works' is not good enough. We need to work from the definition of combinatorial numbers to prove this special case:

$${}^5C_2 + {}^5C_3 = \frac{5!}{(5-2)!2!} + \frac{5!}{(5-3)!3!} \text{ which has to be equal to:}$$

$${}^6C_3 = \frac{6!}{(6-3)!3!}$$

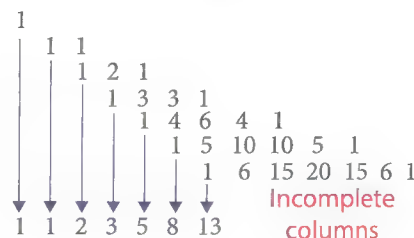
There is some fairly intricate LCM work involved here as you are working with numbers in factorial form. Remember to notice how this works as, to complete the proof, you will need to tackle the general case.

Further Patterns

The rows of Pascal's Triangle appear to sum to powers of 2. Can you prove that they all do?

The first few triangle numbers (1, 3, 6, 10, 15,...) appear in two of the diagonals. Is this a pattern that continues?

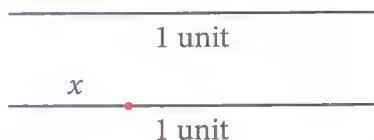
If Pascal's Triangle is written in echelon form, it appears that the Fibonacci sequence emerges.



Does this pattern continue. Can you prove it?

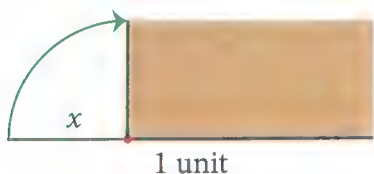
There are many surprising patterns in the Fibonacci Sequence. For example, the limit of the ratio of successive terms is related to the Golden Mean

The Golden Mean



Place a dividing point such that the ratio of the shorter part to the longer part is the same as the ratio of the longer part to the whole.

This becomes the basis of the 'perfect rectangle'.



The Fibonacci Limit

If the n th Fibonacci Number is denoted by F_n , find:

$$\lim_{n \rightarrow \infty} \left(\frac{F_{n+1}}{F_n} \right)$$

Can you prove the relationship between this and the Golden Mean?

SECTION TWO

FUNCTIONS



B.2 Functions

SL 2.5

Basic operations and composite functions

For any two real functions $f: d_f \rightarrow \mathbb{R}, y = f(x)$ and $g: d_g \rightarrow \mathbb{R}, y = g(x)$, defined over domains d_f and d_g respectively, the following rules of algebra apply:

Addition

$(f + g)(x) = f(x) + g(x), d_{f+g} = d_f \cap d_g$ for example:

$$f(x) = \sqrt{x}, x \geq 0 \text{ and } g(x) = x, x \in \mathbb{R}$$

$$(f + g)(x) = \sqrt{x} + x, x \geq 0$$

The domain of the sum is the intersection of the two functions that make up the sum - both functions have to 'work' for the sum to exist.

Subtraction

$(f - g)(x) = f(x) - g(x), d_{f-g} = d_f \cap d_g$

$$f(x) = \sqrt{x}, x \geq 0 \text{ and } g(x) = x, x \in \mathbb{R}$$

$$(f - g)(x) = \sqrt{x} - x, x \geq 0$$

Multiplication

$(f \times g)(x) = f(x) \times g(x), d_{f \times g} = d_f \cap d_g$

$$f(x) = \sqrt{x}, x \geq 0 \text{ and } g(x) = x, x \in \mathbb{R}$$

$$(f \times g)(x) = \sqrt{x} \times x, x \geq 0$$

Division

$$(f \div g)(x) = f(x) \div g(x)$$

The basic domain is, as before: $d_{f \div g} = d_f \cap d_g$. However, we must also exclude cases in which we would be dividing by zero. Thus, values in the domain for which $g(x) = 0$ must also be excluded.

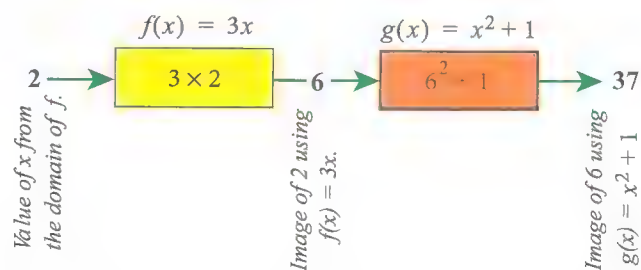
For example: $h(x) = \sqrt{x+1}, x \geq -1$ and $i(x) = x, x \in \mathbb{R}$

$$(g \div i)(x) = \frac{\sqrt{x+1}}{x} \times x, x \geq -1, x \neq 0$$

Composite functions

We now investigate another way in which we can combine functions, namely **composition**.

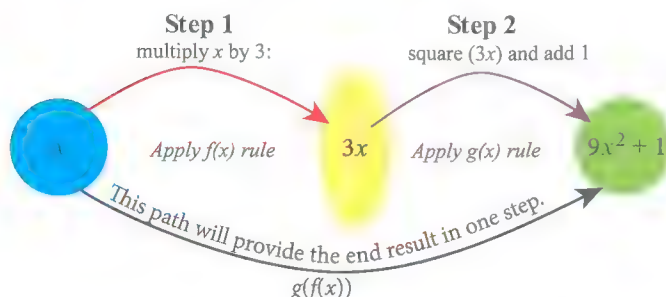
Consider the two functions $f(x) = 3x$ and $g(x) = x^2 + 1$. Observe what happens to the value $x = 2$ as we first apply the function $f(x)$ and then the function $g(x)$ to the image of the first mapping, i.e.



Such a combination of functions leads to the question "Is there a third function that will enable us to produce the same result in one step?"

We consider any value x that belongs to the domain of f and follow 'its path':

- 1 This value of x , has as its image the value $f(x) = 3x$.
- 2 The resulting number, $3x$, now represents an element of the domain of g .
- 3 The image of $3x$ under the mapping g is given by $g(3x) = (3x)^2 + 1 = 9x^2 + 1$.



We can now test our result by using the value of $x = 2$ with the mapping $x \mapsto 9x^2 + 1$.

For $x = 2$, we have $9(2)^2 + 1 = 9 \times 4 + 1 = 37$, which agrees with our previous result.

The two **critical steps** in this process are:

- 1 That the image under the first mapping must belong to the domain of the second mapping.
- 2 The expression $g(f(x))$ exists.

Notation

The expression $g(f(x))$ is called the composite function of f and g and is denoted by $g \circ f$.

Example B.2.1

Given the functions:

$$f(x) = x^2 + 1 \quad \text{and} \quad g(x) = \ln(x - 1),$$

find the composite function $(g \circ f)(x)$.

The composite function $(g \circ f)(x) = g(f(x))$

$$= \ln(f(x) - 1) = \ln(x^2 + 1 - 1) = \ln x^2$$

Example B.2.2

Given the functions $f(x) = 2 - x$ and $g(x) = \sqrt{x - 1}$, find the composite function $(g \circ f)(x)$.

The composite function $(g \circ f)(x) = g(f(x))$

$$= \sqrt{f(x) - 1} = \sqrt{(2 - x) - 1} = \sqrt{1 - x}$$

In the previous example, we have:

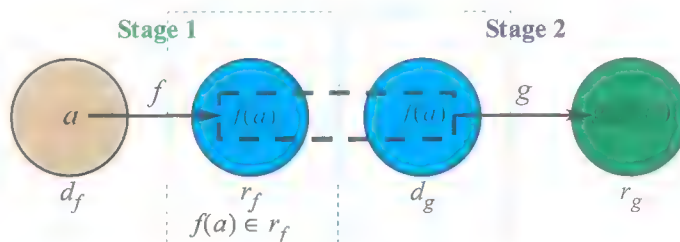
$$(g \circ f)(-1) = \sqrt{1 - (-1)} = \sqrt{2} \text{ but}$$

$$(g \circ f)(2) = \sqrt{1 - 2} = \sqrt{-1} \text{ is undefined!}$$

So what went wrong? To answer this question let's take another look at the composition process.

The process is made up of two stages:

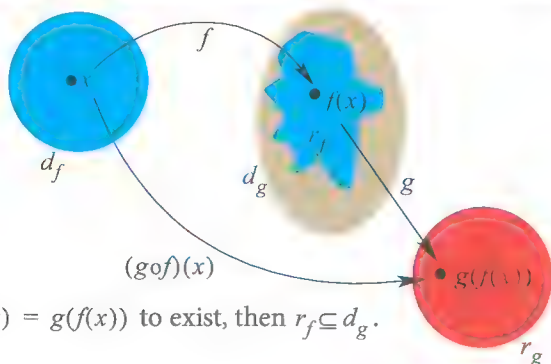
- Stage 1:** An element from the domain of the first function, $f(x)$ is used to produce an image. That is, using $x = a$ we produce the image $f(a)$.
- Stage 2:** Using the second function, $g(x)$, the image, $f(a)$, is used to produce a second image $g(f(a))$.



From the diagram, the result of stage 1 is $f(a)$ (which belongs to the range of f) we also observe that at stage 2, when using the value $f(a)$ (produced from stage 1) we have assumed that $f(a)$ belongs to the domain of $g(x)$. This is where problems can arise – as seen in Example B.2.2.

To overcome this difficulty we need to strengthen our definition of composition of functions as well as ensuring the existence of composite functions.

What we need to prove is that all values produced from stage 1, i.e. $f(a)$, are values in the domain of the function in stage 2, i.e. $f(a) \in d_g$. Making use of a mapping diagram, we show the inter-relation between the range of f , r_f , and the domain of g , d_g .



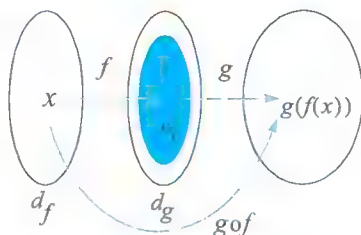
For $(g \circ f)(x) = g(f(x))$ to exist, then $r_f \subseteq d_g$.

What is the domain of $g \circ f$?

If we refer to the diagram alongside, we see that

if $r_f \subseteq d_g$, then $d_{g \circ f} = d_f$.

This means that we can substitute values of x that belong to the domain of f directly into the expression $g(f(x))$ (once we have established that it exists).



Example B.2.3

If $f(x) = \sqrt{x+1}$, $x \in (0, \infty)$ and $g(x) = x^3$, $x \in \mathbb{R}$, determine if $g \circ f$ exists, and find an expression for $g \circ f$ if it does.

For $g \circ f$ to exist we must have that $r_f \subseteq d_g$.

Using a calculator we obtain the range of f from its sketch, in this case, $r_f = (1, \infty)$.

The domain of g is $(-\infty, \infty)$ (i.e. the real field).

Then, given that $(1, \infty) \subseteq (-\infty, \infty)$, $g \circ f$ does exist.

We are now able to determine $g \circ f$.

First we determine the equation $g(f(x))$:

$$g(f(x)) = g(\sqrt{x+1}) = (\sqrt{x+1})^3 = (x+1)^{3/2}.$$

Next we need the domain of $g \circ f$.

As we have seen, $d_{g \circ f} = d_f$, $\therefore d_{g \circ f} = (0, \infty)$.

Therefore, $g \circ f: (0, \infty) \rightarrow \mathbb{R}$, $(g \circ f)(x) = (x+1)^{3/2}$.

Hint on setting out

When solving problems that involve the use of composition, it is useful to set up a **domain-range table** in order to help us determine the existence of the composition. Such a table includes information about the domain and range of both the functions under consideration:

	domain	range
f	d_f	r_f
g	d_g	r_g

The existence of $g \circ f$ can then be established by looking at r_f and d_g . Similarly, the existence of $f \circ g$ can be established by comparing r_g and d_f .

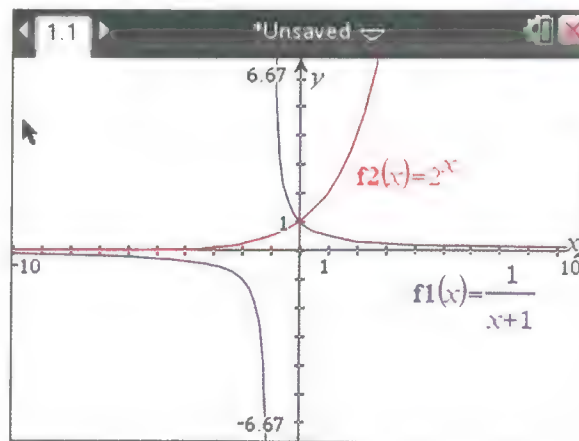
Example B.2.4

Find $g \circ f$ and its range, given that:

$$g(x) = \frac{1}{x+1}, x \in \mathbb{R} \setminus \{-1\} \text{ and } f(x) = 2^x, x \in \mathbb{R}.$$

We first sketch the graphs of both functions to help us complete the domain-range table:

$$g(x) = \frac{1}{x+1}, x \in \mathbb{R} \setminus \{-1\}, \quad f(x) = 2^x, x \in \mathbb{R}.$$



We now complete the table:

	domain	range
f	$\mathbb{R}$	$]0, \infty[$
g	$\mathbb{R} \setminus \{-1\}$	$\mathbb{R} \setminus \{0\}$

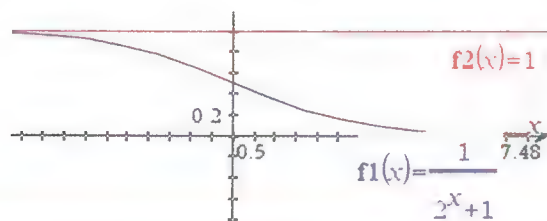
Using the table we see that $r_f \subseteq d_g \Rightarrow g \circ f$ exists.

We can now determine $g \circ f$: $g(f(x)) = \frac{1}{f(x)+1} = \frac{1}{2^x+1}$.

We also have that $d_{g \circ f} = d_f$, $\therefore d_{g \circ f} = \mathbb{R}$.

Therefore, $g \circ f : \mathbb{R} \rightarrow \mathbb{R}$, where $(g \circ f)(x) = \frac{1}{2^x + 1}$.

Making use of a calculator we see that the range of $g \circ f$ is $]0, 1[$.



Example B.2.5

Given that $f(x) = \sqrt{x-1}$ and $g(x) = \ln(x)$. Does $f \circ g$ exist? If so, define fully the function $f \circ g$. If not, find a suitable restriction on the domain of g so that $f \circ g$ exists.

For $f \circ g$ to exist it is necessary that $r_g \subseteq d_f$. To determine the range of g we need to know the domain of g . Using the implied domain we have that $d_g =]0, \infty[$ and so, $r_g = \mathbb{R}$.

However, the implied domain of f is $[1, \infty[$. Then, as $r_g \not\subseteq d_f$, $f \circ g$ does not exist.

In order that $f \circ g$ exists we need to have $rg \subseteq [1, \infty[$, i.e. we must have that $g(x) \geq 1$. What remains then, is to find those values of x such that $g(x) \geq 1$.

Now, $g(x) = \ln x$ therefore, $g(x) \geq 1 \Leftrightarrow \ln x \geq 1 \Leftrightarrow x \geq e$.

So, if the domain of g is restricted to $[e, \infty[$ or any subset of $[e, \infty[$, then $f \circ g$ will exist.

Does $g \circ f = f \circ g$?

In general the answer is **no**! However, there exist situations when $(f \circ g)(x) = (g \circ f)(x)$ – we will look at such cases in the next section.

Consider Example B.2.4, where $g(x) = \frac{1}{x+1}$, $x \in \mathbb{R} \setminus \{-1\}$ and $f(x) = 2^x$, $x \in \mathbb{R}$.

From our previous working, we have that $(g \circ f)(x) = \frac{1}{2^x + 1}$.

To determine if $(f \circ g)(x)$ exists, we will need to determine if $r_g \subseteq d_f$. Using the domain-range table we have that $r_g = \mathbb{R} \setminus \{0\}$ and $d_f = \mathbb{R}$. Therefore as $r_g \subseteq d_f \Rightarrow f \circ g$ does exist.

We then have, $(f \circ g)(x) = f(g(x)) = 2^{g(x)} = 2^{\frac{1}{x+1}}$.

To determine the domain of $f \circ g$, we use the fact that, $d_{f \circ g} = d_g$ so that $d_g = \mathbb{R} \setminus \{-1\}$.

Then, $f \circ g : \mathbb{R} \setminus \{-1\} \rightarrow \mathbb{R}$, where $(f \circ g)(x) = 2^{\frac{1}{x+1}}$.

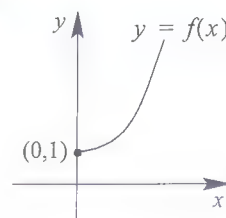
In this case, $(f \circ g)(x) \neq (g \circ f)(x)$.

Example B.2.6

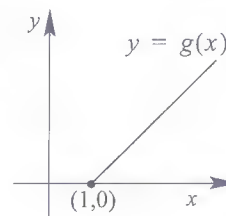
Given $f(x) = x^2 + 1$ where $x \geq 0$ and $g(x) = x - 1$ where $x \geq 0$, determine the functions $(f \circ g)(x)$ and $(g \circ f)(x)$ (if they exist). For the composite functions that exist, find the image of $x = 3$.

We first set up the domain-range table:

	domain	range
f	$[0, \infty)$	$[1, \infty)$
g	$[1, \infty)$	$[0, \infty)$



From the table we see that $r_g \subseteq d_f$ and $r_f \subseteq d_g$, and so, both $(f \circ g)(x)$ and $(g \circ f)(x)$ exist. We can now determine both composite functions.



We start with $(f \circ g)(x)$:

$$(f \circ g)(x) = f(g(x)) = (g(x))^2 + 1 = (x - 1)^2 + 1$$

As $d_{f \circ g} = d_g = [1, \infty[$, we have that:

$$(f \circ g)(x) = (x - 1)^2 + 1, \text{ where } x \geq 1$$

Next, we find $(g \circ f)(x)$:

$$(g \circ f)(x) = g(f(x)) = f(x) - 1 = (x^2 + 1) - 1 = x^2$$

Also, $d_{g \circ f} = d_f = [0, \infty[$, and so, $(g \circ f)(x) = x^2$, where $x \geq 0$

To find the image of 3, we substitute $x = 3$ into the final equations.

$$(f \circ g)(3) = (3 - 1)^2 + 1 = 2^2 + 1 = 5,$$

$$\text{whereas, } (g \circ f)(3) = (3)^2 = 9.$$

Exercise B.2.1

1. Fully define the functions:

a $f + g$ b fg given that :

i $f(x) = x^2$ and $g(x) = \sqrt{x}$

ii $f(x) = \ln x$ and $g(x) = \frac{1}{x}$

iii $f(x) = \sqrt{9 - x^2}$ and $g(x) = \sqrt{x^2 - 4}$

Find the range for case a.

2. Fully define the functions:

a $f - g$ b f/g

(and find the range for case a) given that :

i $f(x) = e^x$ and $g(x) = 1 - e^x$

ii $f(x) = x + 1$ and $g(x) = \sqrt{x + 1}$

iii $f(x) = |x - 2|$ and $g(x) = |x + 2|$

3. All of the following functions are mappings of $\mathbb{R} \rightarrow \mathbb{R}$ unless otherwise stated.

a Determine the composite functions $(f \circ g)(x)$ and $(g \circ f)(x)$, if they exist.

b For the composite functions in part a that do exist, find their range.

i $f(x) = x + 1, g(x) = x^3$

ii $f(x) = x^2 + 1, g(x) = \sqrt{x}, x \geq 0$

iii $f(x) = (x + 2)^2, g(x) = x - 2$

iv $f(x) = \frac{1}{x}, x \neq 0, g(x) = \frac{1}{x}, x \neq 0$

v $f(x) = x^2, g(x) = \sqrt{x}, x \geq 0$

vi $f(x) = x^2 - 1, g(x) = \frac{1}{x}, x \neq 0$

vii $f(x) = \frac{1}{x}, x \neq 0, g(x) = \frac{1}{x^2}, x \neq 0$

4. Given the functions $f: x \mapsto 2x + 1, x \in]-\infty, \infty[$ and $g: x \mapsto x + 1, x \in]-\infty, \infty[$. Find the functions:

a $(f \circ g)$ b $(g \circ f)$ c $(f \circ f)$

5. Determine the function g , given that $f: x \mapsto x + 1, x \in \mathbb{R}$ and $g \circ f: x \mapsto x^2 + 2x + 2, x \in \mathbb{R}$.

6. The functions f and g are defined by $f: x \mapsto x + 1, x \in \mathbb{R}$ and $g: x \mapsto x + \frac{1}{x}, x \in \mathbb{R} \setminus \{0\}$.

Find the composite functions (where they exist) of the following, stating the range in each case.

a $f \circ g$ b $g \circ f$ c $g \circ g$

7. If $g: x \mapsto x^3 + 1, x \in \mathbb{R}$ and $f: x \mapsto \sqrt{x}, x \in [0, \infty[$, find:

a $(g \circ f)(4)$ b $(f \circ g)(2)$

8. Given that $f: x \mapsto x + 5, x \in \mathbb{R}$ and $h: x \mapsto x - 7, x \in \mathbb{R}$, show that $(f \circ h)(x)$ is equal to $(h \circ f)(x)$ for all $x \in \mathbb{R}$.

9. If $f(x) = 4x + 1$ and $g(f(x)) = 16x^2 + 8x + 4$, find $g(x)$.

10. If $f(x) = 2x + 1$ and $g(f(x)) = \frac{1}{2}\sqrt{4x^2 + 4x} + 2$, find $g(x)$.

11. Solve the equation $(f \circ g)(x) = 0$, where:
- a $f: x \mapsto x + 5, x \in \mathbb{R}$ and $g: x \mapsto x^2 - 6, x \in \mathbb{R}$.
- b $f: x \mapsto x^2 - 4, x \in \mathbb{R}$ and $g: x \mapsto x + 1, x \in \mathbb{R}$.
12. Given that $f: x \mapsto 2x + 1, x \in \mathbb{R}$, determine the two functions g , given that:
- a $(g \circ f)(x) = \frac{1}{2x + 1}$
- b $(f \circ g)(x) = \frac{1}{2x + 1}$



Extra questions

Identity and Inverse Functions

Before we start our discussion of inverse functions, it is worthwhile looking back at a fundamental area of algebra – algebraic operations. The relevant algebraic properties for real numbers $a \in \mathbb{R}$, $b \in \mathbb{R}$ and $c \in \mathbb{R}$ are:

	Under addition
Closure	$a + b \in \mathbb{R}$
Commutativity	$a + b = b + a$
Associativity	$(a + b) + c = a + (b + c)$
Existence of the identity	$0: a + 0 = 0 + a = a$
Inverse element	$-a: a + (-a) = 0 = (-a) + a$

	Under multiplication
Closure	$a \times b \in \mathbb{R}$
Commutativity	$a \times b = b \times a$
Associativity	$(a \times b) \times c = a \times (b \times c)$
Existence of the identity	$1: a \times 1 = 1 \times a = a$
Inverse element	$\frac{1}{a}: \left(\frac{1}{a}\right) \times a = 1 = a \times \left(\frac{1}{a}\right)$

Just as there exists an **identity element** for addition, i.e. 0 and for multiplication, i.e. 1 under the real number system, it seems reasonable to assume that an identity element exists when dealing with functions.

It should be noted that without an identity element, equations such as $x + 2 = 7$ and $2x = 10$ could not be solved under the real number system. Because we take the process of solving these equations for granted, sometimes we lose sight of the underpinning algebraic process that led to their solution.

For example, to solve $x + 2 = 7$ we would write $x = 5$ as the next step. However, if we break the process down we have the following:

$$x + 2 = 7 \Leftrightarrow x + 2 + (-2) = 7 + (-2) \text{ (Inverse element)}$$

$$\Leftrightarrow x + 0 = 5 \text{ (Identity)}$$

$$\Leftrightarrow x = 5$$

So that without the identity element, we would not be able to make the last statement!

Consider the two functions $f: x \mapsto \mathbb{R}, \sqrt[3]{x}, x \in \mathbb{R}$ and $g: x \mapsto \mathbb{R}, x^3, x \in \mathbb{R}$. The composite functions $f \circ g$ and $g \circ f$ exist. The composite functions are then given by:

$$(f \circ g)(x) = f(g(x)) = \sqrt[3]{g(x)} = \sqrt[3]{x^3} = x \text{ and}$$

$$(g \circ f)(x) = g(f(x)) = (f(x))^3 = (\sqrt[3]{x})^3 = x.$$

For this particular example we have the result that:

$$f(g(x)) = x = g(f(x)).$$

Using an analogy to the algebraic properties for the real number system, we introduce the identity function.

Identity function

We define the identity function, I , as $I(x) = x$

If an identity function, I , exists then it must have the property that for any given function f

$$f \circ I = I \circ f = f \text{ or } f(I(x)) = I(f(x)) = f(x)$$

As we have already seen, it is found that a function with the rule $I(x) = x$ has the required properties. However, unlike its counterpart in the real number system, where the identities are unique, the domain of the identity function is chosen to match that of the function f .

For example, if $f(x) = \sqrt{x}$, $x \geq 0$ then $I(x) = x$ where $x \geq 0$.

Whereas if $f(x) = 2x$, $x \in]-\infty, \infty[$ then $I(x) = x$ where $x \in]-\infty, \infty[$.

The existence of the identity function leads us to investigate the existence of an **inverse function**.

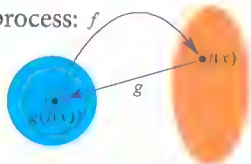
Inverse function

Using an analogy to the real number system, the concept of an inverse requires that given some function f , there exists an inverse function, f^{-1} , such that

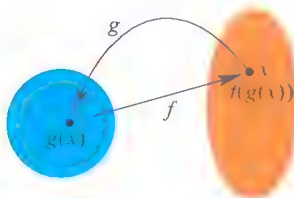
$$f \circ f^{-1} = I = f^{-1} \circ f \text{ or } f(f^{-1}(x)) = x = f^{-1}(f(x))$$

The ' -1 ' used in f^{-1} should not be mistaken for an exponent, i.e. $f^{-1}(x) \neq \frac{1}{f(x)}$!

Looking back at the two functions $f: x \mapsto \mathbb{R}, \sqrt[3]{x}, x \in \mathbb{R}$ and $g: x \mapsto \mathbb{R}, x^3, x \in \mathbb{R}$ we notice that the function g is the reverse operation of function f , and function f is the reverse operation of function g . Making use of a mapping diagram we can 'visualise' the process:



1. Use an element (x) from the domain of the function f and obtain its image $f(x)$.
2. Using this image, which must be an element of the domain of g , we then apply g to $f(x)$ and obtain its image, $g(f(x))$ resulting in the value x that we started with.



1. Use an element (x) from the domain of the function g and obtain its image $g(x)$.
2. Using this image, which must be an element of the domain of f , we then apply f to $g(x)$ and obtain its image, $f(g(x))$ resulting in the value x that we started with.

We can now make some observations:

1. The domain of f , d_f , must equal the range of g , r_g and the domain of g , d_g must equal the range of f , r_f .

2. For the uniqueness of ' x ' to be guaranteed both f and g must be **one-to-one functions**.

We therefore have the result:

If f and g are **one-to-one functions**, such that $f(g(x)) = x = g(f(x))$, then g is known as the inverse of f and f as the inverse of g .

In our case we write, $g(x) = f^{-1}(x)$ and $f(x) = g^{-1}(x)$. This then brings us back to the notation we first introduced for the inverse function. We summarise our findings:

For the inverse function of f , f^{-1} (read as f inverse) to exist, then:

1. f must be a one to one function.
2.
 - i the domain of f is equal to the range of f^{-1} , i.e. $d_f = r_{f^{-1}}$
 - ii the range of f is equal to the domain of f^{-1} , i.e. $d_{f^{-1}} = r_f$
3. $f(f^{-1}(x)) = x = f^{-1}(f(x))$.

How do we find the Inverse Function?

A guideline for determining the inverse function can be summarized as follows:

Step 1: Check that the function under investigation is a one-to-one function. This is best done by using a sketch of the function.

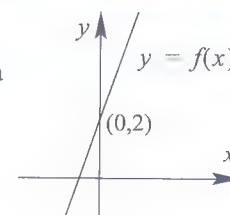
Step 2: Use the expression $f(f^{-1}(x)) = x$ to solve for $f^{-1}(x)$.

Step 3: Use the fact that $d_{f^{-1}} = r_f$, and $d_f = r_{f^{-1}}$ to complete the problem.

Example B.2.7

Find the inverse of the function:

$$f: x \mapsto 5x + 2, \text{ where } x \in \mathbb{R}$$



We start by checking if the function is a one-to-one function.

From the graph, it is clearly the case that $f(x)$ is a one-to-one function.

Making use of the result that $f(f^{-1}(x)) = x$ to solve for $f^{-1}(x)$

$$f(f^{-1}(x)) = 5f^{-1}(x) + 2$$

$$\text{Then, } f(f^{-1}(x)) = x \Rightarrow 5f^{-1}(x) + 2 = x$$

$$\Leftrightarrow 5f^{-1}(x) = x - 2$$

$$\Leftrightarrow f^{-1}(x) = \frac{1}{5}(x - 2)$$

To complete the question we need the domain of f^{-1} . We already know that $d_{f^{-1}} = r_f$, therefore all we now need is the range of f , so $d_{f^{-1}} = \mathbb{R}$ (using the graph of $f(x)$).

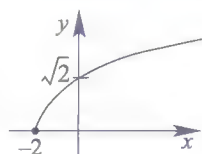
That is, $f^{-1}: \mathbb{R} \rightarrow \mathbb{R}$ where $f^{-1}(x) = \frac{1}{5}(x - 2)$.

Example B.2.8

Find the inverse function of $f(x) = \sqrt{x+2}, x \geq -2$ and sketch its graph.

A quick sketch of the graph of f verifies that it is a one-to-one function.

We can now determine the inverse function, $f^{-1}(x)$:



Now, $f(f^{-1}(x)) = \sqrt{f^{-1}(x)+2}$, so, using $f(f^{-1}(x)) = x$ we have:

$$\sqrt{f^{-1}(x)+2} = x \Leftrightarrow f^{-1}(x)+2 = x^2 \quad (\text{squaring both sides})$$

$$\Leftrightarrow f^{-1}(x) = x^2 - 2$$

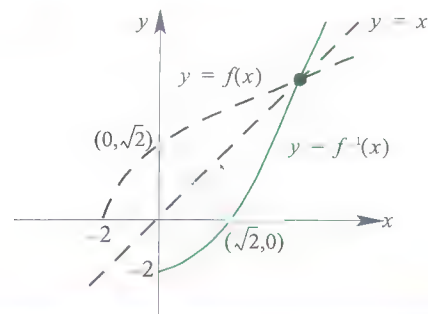
To fully define f^{-1} we need to determine the domain of f^{-1} .

We do this by using the result that $d_{f^{-1}} = r_f$.

Using the graph shown above, we have that $r_f = [0, \infty[$ $\therefore d_{f^{-1}} = [0, \infty[$. We are now in a position to fully define the inverse function.

We have, $f^{-1}: [0, \infty) \rightarrow \mathbb{R}, f^{-1}(x) = x^2 - 2$.

We can now sketch the graph of the inverse function:

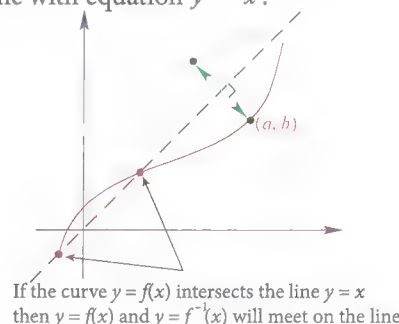


We make two important observations:

1. The graph of $y = f^{-1}(x)$ is the graph of $y = f(x)$ reflected about the line $y = x$.
2. Points of intersection of the graphs $y = f^{-1}(x)$ and $y = f(x)$ will always occur where both curves meet the line $y = x$.

The relationship between the graph of a function f and the graph of its inverse function f^{-1} can be explained rather neatly, because all that has actually happened is that we have interchanged the x - and y -values, i.e. $(a, b) \leftrightarrow (b, a)$. In doing so, we find that $d_{f^{-1}} = r_f$, and $d_f = r_{f^{-1}}$.

This then has the result that the graph of the inverse function $f^{-1}(x)$ is a reflection of the graph of the original function $f(x)$ about the line with equation $y = x$.



Graphing the inverse function

Because of the nature of the functions $f^{-1}(x)$ and $f(x)$, when finding the points of intersection of the two graphs, rather than solving $f^{-1}(x) = f(x)$, it might be easier to solve the equations $f^{-1}(x) = x$ or $f(x) = x$. The only caution when solving the latter two, is to always keep in mind the domain of the original function.

One interesting function is $f(x) = \frac{1}{x}, x \neq 0$.

It can be established that its inverse function is given by:

$$f^{-1}(x) = \frac{1}{x}, x \neq 0$$

$$\text{i.e. } f(f^{-1}(x)) = x \Rightarrow \frac{1}{f^{-1}(x)} = x \Leftrightarrow f^{-1}(x) = \frac{1}{x}.$$

Then, as the two functions are identical, it is its **self-inverse**.

Sketching the graph of $f(x) = \frac{1}{x}$, $x \neq 0$ and reflecting it about the line $y = x$ will show that the two graphs overlap each other. Note then, that self-inverse functions also intersect at points other than just those on the line $y = x$ – basically because they are the same functions!

There is a second method that we can use to find the inverse of a function. The steps required are:

Step 1: Let y denote the expression $f(x)$.

Step 2: Interchange the variables x and y .

Step 3: Solve for y .

Step 4: The expression in step 3 gives the inverse function, $f^{-1}(x)$.

We work through an example using this method.

Example B.2.9

Find the inverse function, f^{-1} of $f:]1, \infty[\rightarrow \mathbb{R}$ where $f(x) = \frac{2}{x-1} + 1$ and sketch its graph.

Let $y = f(x)$, giving $y = \frac{2}{x-1} + 1$.

Next, we interchange the variables x and y : $x = \frac{2}{y-1} + 1$.

We now solve for y : $x = \frac{2}{y-1} + 1 \Leftrightarrow x-1 = \frac{2}{y-1}$

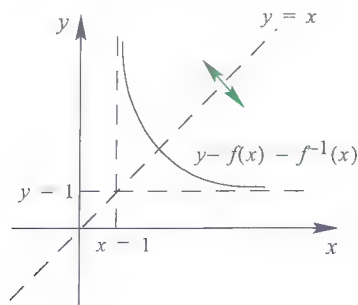
$$\Leftrightarrow \frac{1}{x-1} = \frac{y-1}{2} \quad (\text{inverting both sides})$$

$$\Leftrightarrow y-1 = \frac{2}{x-1}$$

Therefore, we have that $y = \frac{2}{x-1} + 1$

That is, $f^{-1}(x) = \frac{2}{x-1} + 1$, $x > 1$.

We notice that the original function $y = f(x)$ and its inverse



function $y = f^{-1}(x)$ are the same. This becomes obvious when we sketch both graphs on the same set of axes.

When reflected about the line $y = x$, they are identical!

Again, we have an example of a self-inverse function.

Example B.2.10

Find the inverse of the function $g: x \mapsto 4^x - 2$, $x \in \mathbb{R}$.

The function $g: x \mapsto 4^x - 2$, $x \in \mathbb{R}$ is a one-to-one function and so its inverse function exists.

Using the result that $g(g^{-1}(x)) = x$ we have:

$$4^{g^{-1}(x)} - 2 = x$$

$$\Leftrightarrow 4^{g^{-1}(x)} = x + 2$$

$$\therefore g^{-1}(x) = \log_4(x+2) \quad (\text{using } N = b^x \Leftrightarrow x = \log_b N)$$

Now, $d_{g^{-1}} = r_g =]-2, \infty[$ (we have obtained the range by using a sketch of $g(x)$)

Therefore the inverse, g^{-1} , is given by

$$g^{-1}: x \mapsto \log_4(x+2), x > -2$$

Example B.2.11

Find the inverse of $g: \left(\frac{1}{2}, \infty\right) \rightarrow \mathbb{R}, x \mapsto \log_{10}(2x-1) + 2$.

This time we make use of the second method of finding the inverse function.

Let $y = \log_{10}(2x-1) + 2$, interchanging x and y , we have:

$$x = \log_{10}(2y-1) + 2 \Leftrightarrow x-2 = \log_{10}(2y-1)$$

$$\Leftrightarrow 2y-1 = 10^{x-2} \quad (\text{using } N = b^x \Leftrightarrow x = \log_b N)$$

$$\Leftrightarrow y = \frac{1}{2}(10^{x-2} + 1).$$

Therefore, we have that $f^{-1}(x) = \frac{1}{2}(10^{x-2} + 1)$.

Next, $d_{f^{-1}} = r_f = (-\infty, \infty)$, so the inverse function is:

$$f^{-1}: x \mapsto \frac{1}{2}(10^{x-2} + 1), x \in (-\infty, \infty).$$

Example B.2.12

Find the inverse function of $f(x) = 4x^2 + 8x - 1, x \geq -1$

$$\text{Let } y = f(x)$$

$$y = 4x^2 + 8x - 1$$

$$= 4(x^2 + 2x) - 1$$

$$= 4(x^2 + 2x + 1) - 1 - 4$$

$$= 4(x+1)^2 - 5$$

Interchanging x and y .

$$x = 4(y+1)^2 - 5$$

$$(y+1)^2 = \frac{x+5}{4}$$

$$y+1 = \pm \frac{\sqrt{x+5}}{2}$$

$$y = \pm \frac{\sqrt{x+5}}{2} - 1$$

Since the original domain is restricted to $x \geq -1$ and the range is $y \geq -5$, the inverse of the original quadratic equation must be restricted to the domain $x \geq -5$ with the range of $y \geq -1$.

$$\text{Hence } f^{-1}(x) = \frac{\sqrt{x+5}}{2} - 1, x \geq -5$$

Exercise B.2.2

1. Find the inverse function for each of the following.

a $f(x) = 2x + 1, x \in \mathbb{R}$

b $f(x) = x^3, x \in \mathbb{R}$

c $g(x) = \frac{1}{3}x - 3, x \in \mathbb{R}$

d $g(x) = \frac{2}{5}x + 2, x \in \mathbb{R}$

e $h(x) = \sqrt{x+1}, x > -1$

f $f(x) = \sqrt{x} + 1, x \geq 0$

g $f(x) = \frac{1}{x+1}, x > -1$

h $h(x) = \frac{1}{\sqrt{x}} - 1, x > 0$

2. Using the graph of the original function, sketch the graph of the corresponding inverse function for each part in Question 1.

3. Find and sketch the inverse function of:

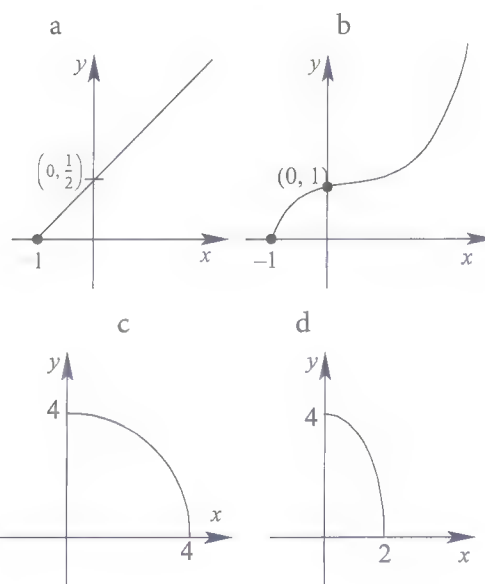
a $f(x) = x^2 - 3, x \geq 0$

b $f(x) = x^2 - 3, x \leq 0$

4. Show that $f(x) = \frac{x}{\sqrt{x^2 + 1}}, x \in \mathbb{R}$ is a one-to-one function.

Hence find its inverse.

5. Sketch the inverse of the following functions.



6. Find the inverse function (if it exists) of the following.

a $f(x) = 3^x + 1, x \in]-\infty, \infty[$

b $f(x) = 2^x - 5, x \in]-\infty, \infty[$

c $f(x) = 3^{2x+1}, x \in]-\infty, \infty[$

d $g(x) = 3 - 10^{x-1}, x \in]-\infty, \infty[$

e $h(x) = \frac{2}{3^x - 1}, x \neq 0$

f $g(x) = \frac{1}{2^x} - 1, x \in]-\infty, \infty[$

7. Using the graph of the original functions in Question 6, sketch the graph of their inverses.

8. Find the inverse function (if it exists) of the following functions:

a $f(x) = \log_2(x+1), x > -1$

b $f(x) = \log_{10}(2x), x > 0$

c $h(x) = 1 - \log_2 x, x > 0$

d $g(x) = \log_3(x-1) - 1, x > 1$

e $h(x) = 2\log_5(x-5), x > 5$

f $f(x) = 2 - \frac{1}{3}\log_{10}(1-x), x < 1$

9. Find the inverse function of $f(x) = x^2 + 2x, x \geq -1$, stating both its domain and range. Sketch the graph of f^{-1} .

10. Find the inverse function of:

a $f(x) = -x + a, x \in]-\infty, \infty[$, where a is real.

b $h(x) = \frac{2}{x-a} + a, x > a$, where a is real.

c $f(x) = \sqrt{a^2 - x^2}, 0 \leq x \leq a$, where a is real.

11. Find the inverse of $h(x) = -x^3 + 2, x \in \mathbb{R}$. Sketch both $h(x)$ and $h^{-1}(x)$ on the same set of axes.

12. Find the largest possible set of positive real numbers S , that will enable the inverse function h^{-1} to exist, given that $h(x) = (x-2)^2, x \in S$.

13. Determine the largest possible positive valued domain, X , so that the inverse function, $f^{-1}(x)$, exists,

given that $f(x) = \frac{3x+2}{2x-3}, x \in X$.

14. a Sketch the graph of $f(x) = x - \frac{1}{x}, x > 0$.

Does the inverse function, f^{-1} exist? Give a reason for your answer.

b Consider the function $g: S \rightarrow \mathbb{R}$ where, $g(x) = x - \frac{1}{x}$

Find the two largest sets S so the inverse function, g^{-1} , exists. Find both inverses and on separate axes, sketch their graphs.

15. Find f^{-1} given that $f(x) = \sqrt{\frac{x}{a} - 1}$ where $a > 1$.

On the same set of axes sketch both the graphs of $y = f(x)$ and $y = f^{-1}(x)$. Find $\{x : f(x) = f^{-1}(x)\}$.

16. Find and sketch the inverse, f^{-1} , of the functions:

a $f(x) = \begin{cases} -\frac{1}{2}(x+1), & x > 1 \\ -x^3, & x \leq 1 \end{cases}$

b $f(x) = \begin{cases} e^{x+1}, & x \leq 0 \\ x+e, & x > 0 \end{cases}$

c $f(x) = \begin{cases} -\ln(x-1), & x > 2 \\ 2-x, & x \leq 2 \end{cases}$

d $f(x) = \begin{cases} \sqrt{x}+4, & x > 0 \\ x+4, & -4 < x < 0 \end{cases}$

17. a On the same set of axes sketch the graph of $f(x) = \frac{1}{a}\ln(x-a), x > a > 0$ and its reciprocal.

b Find and sketch the graph of f^{-1} .

Extra questions



Answers



B.3 Equations and Graphs

SL 2.10

SL 2.11

In this section, we will revisit some general principles of solving equations and drawing graphs. We cover some methods that are specific to particular functions. For example, in the next chapter we will cover the quadratic formula and its use in solving quadratic equations and drawing the graphs of quadratic functions.

This chapter will review more general principles that you can apply in a wide range of situations.

Solving Equations

We referred to the connection between the intersections of graphs and the solutions of equations in Chapter B.3 of the Core text. You might like to review this discussion of the solution of the equation:

$$x^5 - x^2 - 2 = \frac{1}{x+1} + 5$$

Finding a solution of this by rearranging presents difficulties - try it! The solution we are looking for is $x \approx 1.581$

The method is to draw the graphs of

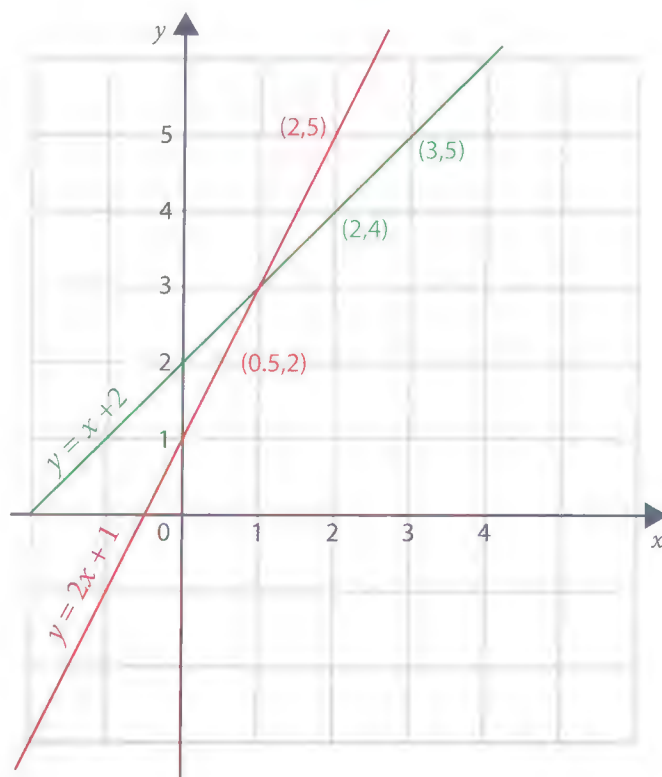
$$y = x^5 - x^2 - 2 \text{ and } y = \frac{1}{x+1} + 5$$



To take a much less complex example, consider the equation: $2x + 1 = x + 2$. This can be solved 'analytically' by subtracting x from both sides to get: $x + 1 = 2$. Subtracting 1 from both sides gives the solution $x = 1$. As we observed above, such solutions are not always available. Then we must fall back on approximate methods of which graphs are one.

To solve $2x + 1 = x + 2$, we plot two graphs:

$$y = 2x + 1 \text{ and } y = x + 2$$



The important thing to realise is that $y = 2x + 1$ is true for every point on the red line. We show (0.5, 2) & (2, 5). $y = x + 2$ is true for each point on the green line. Examples are (2, 4) & (3, 5). The point where the lines intersect, (1, 3), is such that both statements are true simultaneously.

Thus $x = 1$ is a solution to $2x + 1 = x + 2$ and $x = 1, y = 3$ is the solution to the simultaneous equations $y = 2x + 1$ and $y = x + 2$.

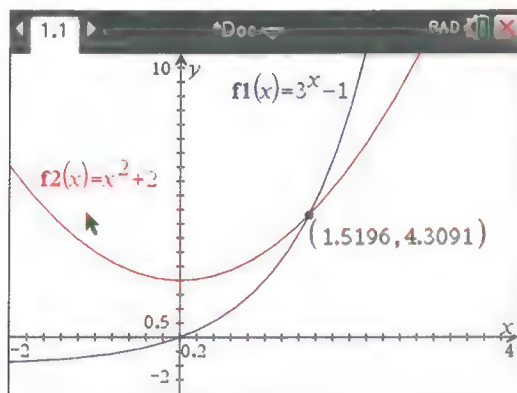
Example B.3.1

Solve, correct to 3 significant figures:

$$3^x - 1 = x^2 + 2, x \geq 0.$$

First, we use a TI calculator. The steps are:

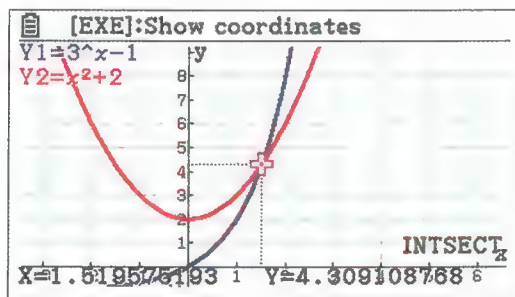
- open a graph document
- enter the functions $f1(x) = 3^x - 1$ and $f2(x) = x^2 + 2$
- adjust the viewing window
- check the accuracy of the answer (MENU, 9-Settings)
- use analyze graph to locate the positive intersection.



The solution is 1.52 to 3 s.f.

Using a Casio calculator:

- open the graph module
- enter the functions $Y1 = 3^x - 1$ and $Y2 = x^2 + 2$
- adjust the viewing window
- use G-Solv, to locate the positive intersection.



The solution is 1.52 to 3 s.f.

Exercise B.3.1

In questions 1-8, find the smallest positive solution to each of the equations, correct to 4 significant figures:

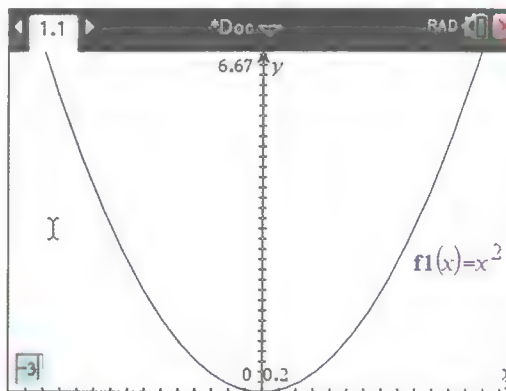
1. $x^3 - 3 = \sqrt{x} + 2$
2. $5 - x^2 = 2^x$
3. $5 - x^3 = x + 1$
4. $\frac{x+1}{x-1} = 2^x$
5. $x^3 - x = x^2$
6. $e^x = 2 - x$
7. $\sqrt{x^2 + 1} = 7 - x^2$
8. $x^x = 3 - x$
9. Find the number of sides of the regular polygon that has an internal angle of 179° .
10. What is the largest integral value of n such that the number of ways in which n people can be seated in a line of n seats is less than a million?

Graphs

We now move on to look at some general principles that can be applied to all graphs.

The general point here is that most graphs have a general shape that is similar for a large class of graph.

Thus, all quadratic functions have parabolic graphs that can be related to the simplest parabola:



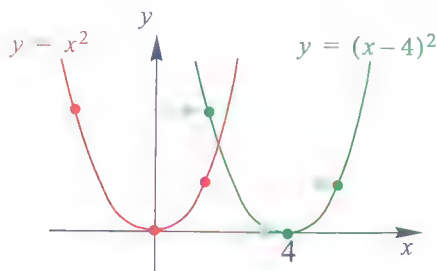
More complex examples can often be seen as transformations of these simpler forms. We look at these next.

Horizontal translation

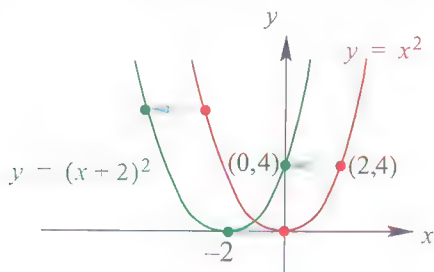
A horizontal translation takes the form: $f(x) \rightarrow f(x-a)$.

We start by looking at the transformation of the basic parabolic graph with equation $y = x^2$. The horizontal translation of $y = x^2$ is given by $y = (x-a)^2$. This transformation represents a **translation along the x-axis**.

For example, the graph of $y = (x-4)^2$ represents the parabola $y = x^2$ translated 4 units to the right ($a = 4$):



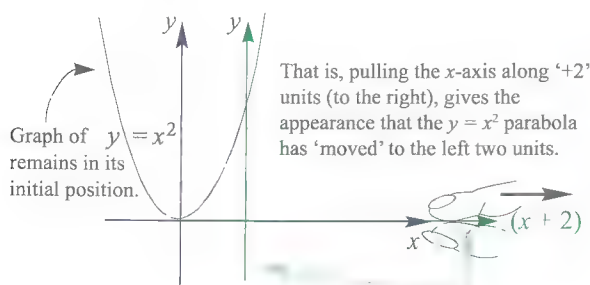
Similarly, the graph of $y = (x+2)^2$ represents the parabola $y = x^2$ translated 2 units to the left ($a = -2$):



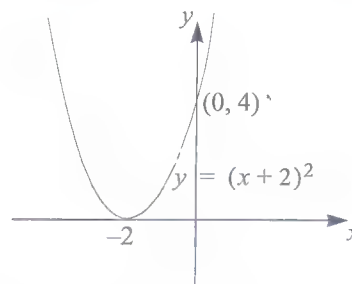
In both situations, it appears as if the graphs have been translated in the 'opposite direction' to the sign of ' a '. That is, $y = (x+2)^2$ has been translated 2 units back (i.e. in the negative direction), while $y = (x-2)^2$ has been translated 2 units forward (i.e. in the positive direction).

The reason for this is **the transformation is applied to the x-values, not the graph**.

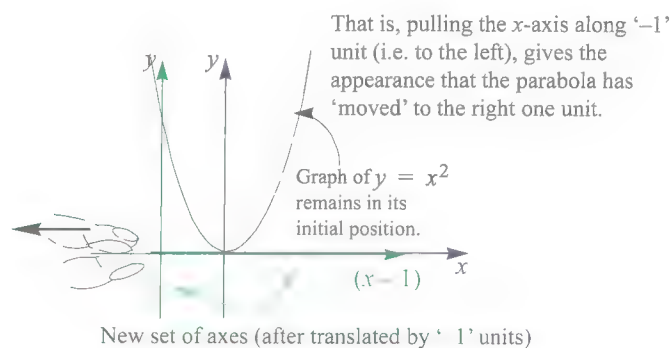
That is, given $y = f(x)$, the graph of $y = f(x+2)$ is telling us to 'Add two units to all the x-values'. In turn, this means, that the combined x/y-axes should be moved in the positive direction by two units (whilst the graph of $y = f(x)$ remains exactly where it is):



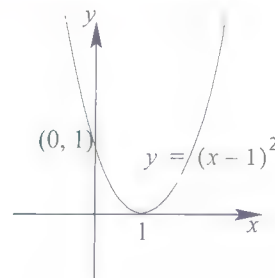
New set of axes (after being translated by '+2' units)



Similarly, given $y = f(x)$, the graph of $y = f(x-1)$ is telling us to 'Subtract one unit from all the x-values'. In turn, this means, that the combined x/y-axes should be moved in the negative direction by one unit (while maintaining the graph of $y = f(x)$ fixed at its original position):



New set of axes (after translated by '-1' units)



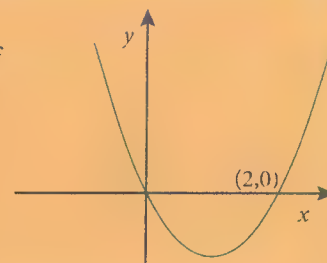
However, whenever we are asked to sketch a graph, rather than drawing the axes after the graph has been sketched, the first thing we do is draw the set of axes and then sketch the graph. This is why there appears to be a sense in which we seem to do the 'opposite' when sketching graphs that involve transformations.

Example B.3.2

Given the graph of $y = f(x)$ sketch the graph of:

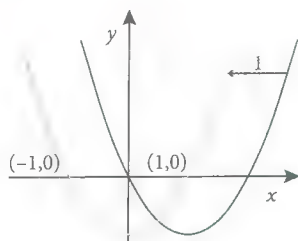
a $y = f(x+1)$

b $y = f\left(x - \frac{1}{2}\right)$



a

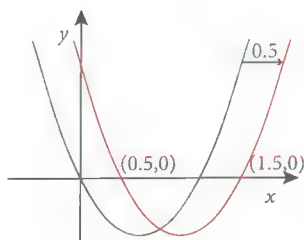
The graph of $y = f(x+1)$ represents a translation along the x -axis of the graph of $y = f(x)$ by 1 unit to the left.



b

The graph of $y = f\left(x - \frac{1}{2}\right)$

represents a translation along the x -axis of the graph of $y = f(x)$ by $\frac{1}{2}$ unit to the right.

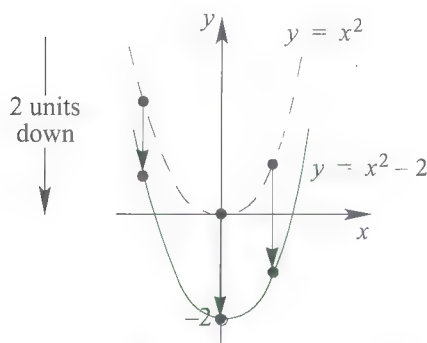


Vertical translation

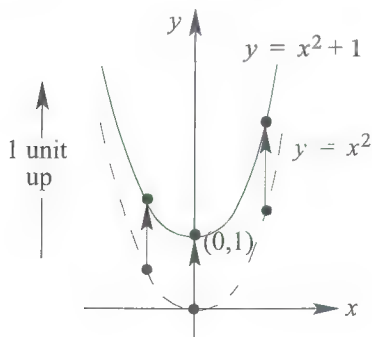
A vertical translation takes on the form $f(x) \rightarrow f(x) + b$.

Again we consider the transformation of the basic parabolic graph with equation $y = x^2$. The vertical translation of $y = x^2$ is given by $y = x^2 + b$. This transformation represents a translation along the y -axis.

For example, the graph of $y = x^2 - 2$ represents the parabola $y = x^2$ translated 2 units down:



Similarly, the graph of $y = x^2 + 1$ represents the parabola $y = x^2$ translated 1 unit up:

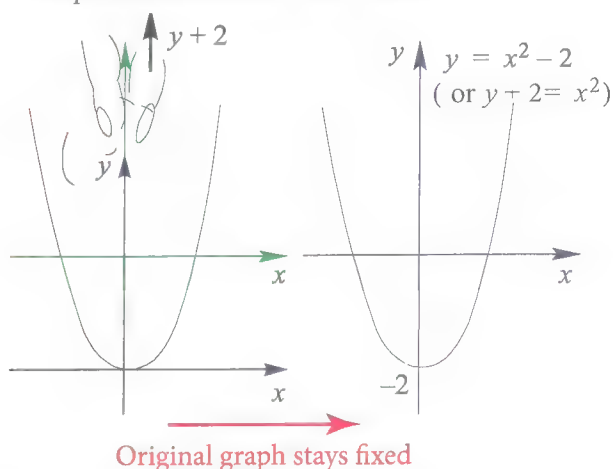


Note that this time, when applying a vertical translation, we are consistent with the sign of b !

The reason is that, although we are sketching the graph of $y = f(x) + b$, we are in fact sketching the graph of $y - b = f(x)$. So that this time, the transformation is applied to the y -values, and not the graph!

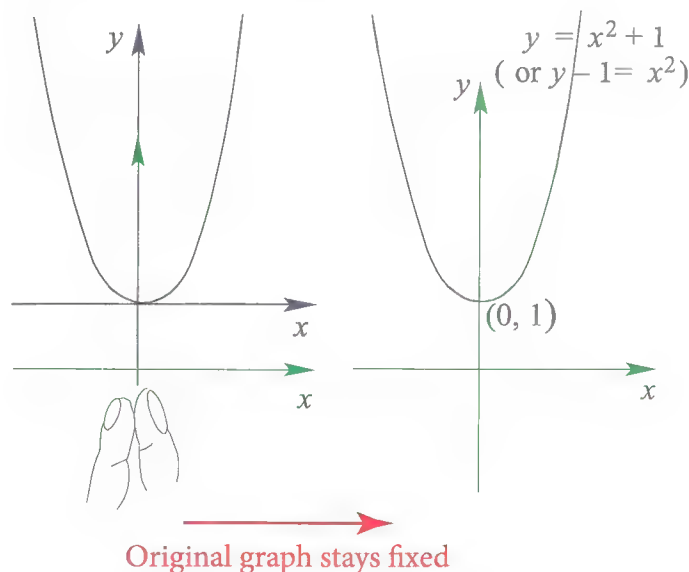
So, if we consider the two previous examples, we have $y = x^2 - 2 \Leftrightarrow y + 2 = x^2$, and so, in this case we would be pulling the y -axis UP 2 units, which gives the appearance that the parabola has been moved down 2 units.

That is, pulling the y -axis along '+2' units (i.e. upwards), gives the appearance that the parabola has 'moved' down 2 units.



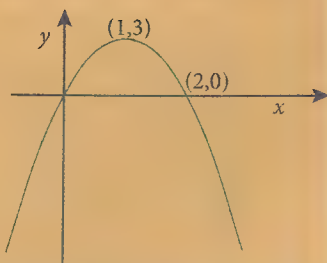
Similarly, $y = x^2 + 1 \Leftrightarrow y - 1 = x^2$, so that in this case we would be pulling the y -axis DOWN 1 unit, which gives the appearance that the parabola has been moved up 1 unit.

That is, pulling the y -axis along '-1' units (i.e. downwards), gives the appearance that the parabola has 'moved' up 1 unit.



Example B.3.3

Given the graph of $y = f(x)$ sketch the graph of:

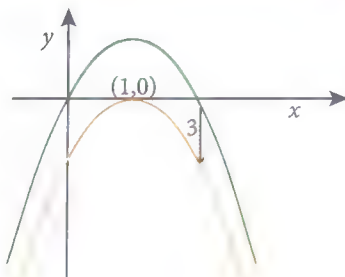


a $y = f(x) - 3$

b $y = f(x) + 2$

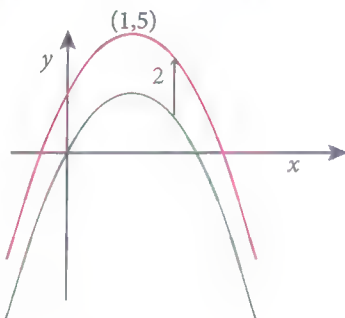
a

The graph of $y = f(x) - 3$ represents a translation along the y -axis of the graph of $y = f(x)$ by 3 units in the downward direction.



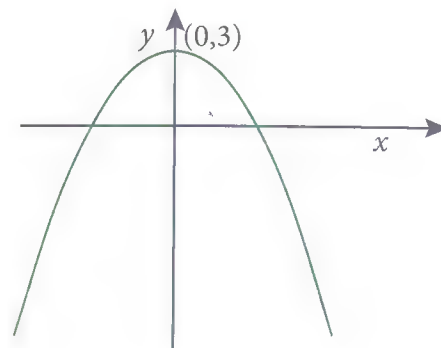
b

The graph of $y = f(x) + 2$ represents a translation along the y -axis of the graph of $y = f(x)$ by 2 units in the upward direction.

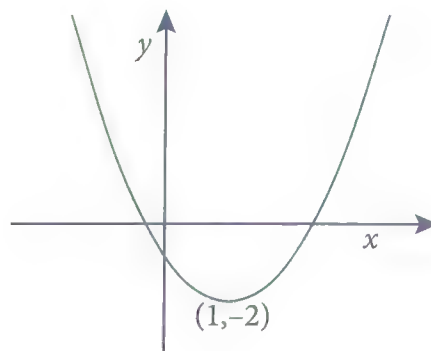


where the graphs of $y = f(x)$ are shown below:

a



b



2. Sketch the following functions and confirm your answers using a graphic calculator.

a $y = (x-1)^2 - 2$

b $y = (x+1)^2 + 1$

c $y = (x-3)^2 - 2$

d $y = -(x-1)^2 - 2$

e $y = (x-2)^2 - 5$

3. State the equations of the lines of symmetry of:

a $y = (x+2)^2 + 7$

b $y = (x-4)^2 - 12$

c $y = -(x+1)^2 + 2$

d $y = (x-12)^2 + 2$

Exercise B.3.2

1. Sketch the graphs of:

i $y = f(x-1)$

ii $y = f(x+1)$

iii $y = f(x-2)$

iv $y = f(x-1)+1$

v $y = f(x+1)-2$

vi $y = f(x-2)+1$

vii $y = f(x+1)+1$

viii $y = f(x-1)-2$

Dilations

Dilation from the x-axis

Other commonly used expressions for dilations from the x -axis are: dilation along the y -axis and dilation parallel to the y -axis – any one of these three expressions can be used when describing this dilation.

The equation $y = p \cdot f(x)$ can be written as $\frac{y}{p} = f(x)$.

We have rearranged the expression so that we can more clearly see the effects that p has on the y -values. That is, the term y/p represents a transformation on the y -axis as opposed to a transformation on the graph of $f(x)$. The effect of ' p ' in the term y/p is that of a **dilation from the x -axis**.

If $|p| > 1$, we *shrink* the y -axis (seeing as we are dividing the y -values by a number larger than one). Whereas if $0 < |p| < 1$ we *stretch* the y -axis.

However, we still need to describe the effect this transformation has on the final appearance of the graph of $f(x)$.

We summarise these results, stating the effects on the graph of $f(x)$:

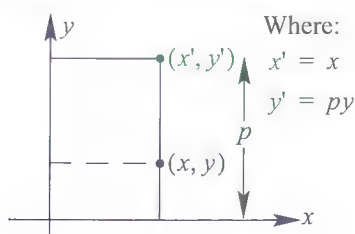
For the curve with equation $y = p \cdot f(x)$,

$|p| > 1$, represents '**stretching**' $f(x)$ by a factor p from the x -axis.

$0 < |p| < 1$, represents '**shrinking**' $f(x)$ by a factor $\frac{1}{p}$ from the x -axis.

A 'stretch' of factor $1/3$ is the same as a 'shrink' of factor 3. However, it is more common that when referring to a dilation from the x -axis, we refer to it as a stretch. So a dilation from the x -axis of factor 3 would imply a stretching effect whereas a dilation from the x -axis of factor $1/3$ would imply a shrinking effect.

The relationship between the original coordinates (x, y) and the new coordinates (x', y') can be seen in the diagram below.



Example B.3.4

Describe the effects on the graph of $y = f(x)$ when the following are sketched.

- a $y' = 2 \cdot f(x)$ b $y = \frac{1}{4} f(x)$
 c $2y = f(x)$

a $y = 2f(x)$ represents a dilation of factor 2 from the x -axis. This means that the graph of $y = f(x)$ would be stretched by a factor of two along the y -axis.

b $y = \frac{1}{4} f(x)$ shows a dilation of factor $1/4$ from the x -axis.

This means that the graph of $y = f(x)$ would shrink by a factor of four along the y -axis.

c $2y = f(x)$ needs to first be written as $y = \frac{1}{2} f(x)$.

This represents a dilation of factor $1/2$ from the x -axis. This means that the graph of $y = f(x)$ would shrink by a factor of two along the y -axis.

Dilation from the y-axis

The equation: $y = f\left(\frac{x}{q}\right)$

represents a transformation applied to the x -values. We now need to consider how this factor ' q ' affects the graph of $f(x)$. The term x/q represents a transformation on the x -values as opposed to a transformation on the graph of $f(x)$. The effect of ' q ' in the term x/q is that of a **dilation from the y -axis**.

If $|q| > 1$, we *shrink* the x -axis (seeing as we are dividing the x -values by a number larger than one). Whereas if $0 < |q| < 1$ we *stretch* the x -axis (because we are dividing by a number less than one but greater than zero). However, we still need to describe the effect this transformation has on the final appearance of the graph of $f(x)$.

We summarise these results, stating the effects on the graph of $f(x)$:

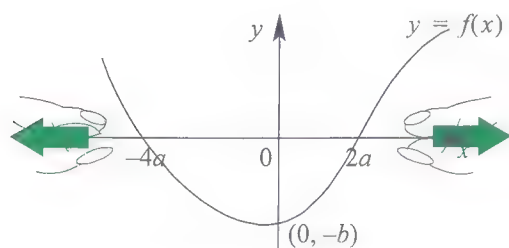
For the curve with equation:

$$y = f\left(\frac{x}{q}\right)$$

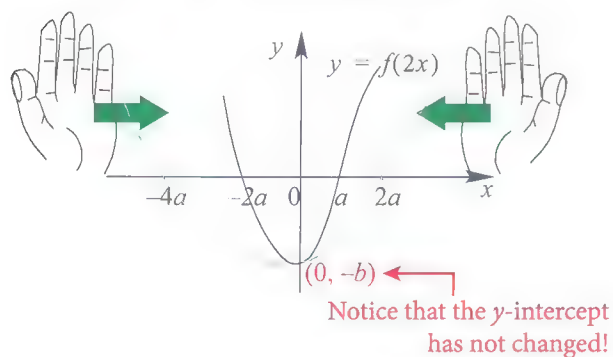
$|q| > 1$, represents 'stretching' $f(x)$ by a factor q from the y -axis.

$0 < |q| < 1$, represents 'shrinking' $f(x)$ by a factor $\frac{1}{q}$ from the y -axis.

So a dilation from the y -axis of factor 2 (e.g. $y = f(x/2)$) would imply a stretching effect whereas a dilation from the y -axis of factor $1/2$ (e.g. $y = f(2x)$) would imply a shrinking effect. We show this in the following diagram.

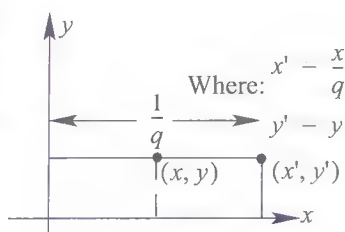


Multiplying the x -values by 2, i.e. stretching the x -axis by a factor of 2 has the same effect as squashing the graph of $y = f(x)$ by a factor of 2.



Again, it must be remembered that the graph of $y = f(x)$ has not changed, it is simply the illusion that the graph has been squashed. That is, relative to the new (stretched) x -axis, the graph of $y = f(x)$ appears to have been squashed.

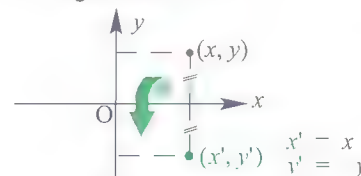
The relationship between the original coordinates (x, y) and the new coordinates (x', y') can be seen in the diagram.



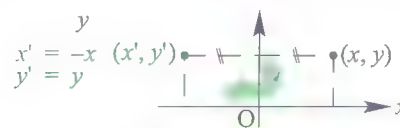
Again, we have no dilation vector to describe this transformation (although there does exist a *dilation matrix*).

Reflections

We first consider reflections about the x -axis and about the y -axis. The effects of reflecting a curve about these axes can be seen in the diagrams below:



When reflecting about the x -axis we observe that the coordinates (x, y) are mapped to the coordinates $(x, -y)$ meaning that the x -values remain the same but the y -values change sign.

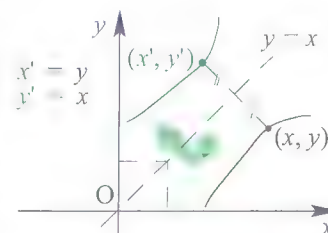


When reflecting about the y -axis we observe that the coordinates (x, y) are mapped to the coordinates $(-x, y)$, meaning that the y -values remain the same but the x -values change sign.

Another type of reflection is the reflection about the line $y = x$, when sketching the inverse of a function.

Reflection about the line $y = x$

When reflecting about the line $y = x$ we observe that the coordinates (x, y) are mapped to the coordinates (y, x) , meaning that the x -values and the y -values are interchanged. If a one-one function undergoes such a reflection, we call its transformed graph the inverse function and denote it by $y = f^{-1}(x)$.



Intercepts

Finding the intercepts of a graph can be very helpful in sketching curves, particularly if, as is the case with parabolas, the curve is symmetric.

y -intercept

The y -intercepts of all graphs are found by substituting $x = 0$ into the function rule.

Thus, $y = x^2 - 4x + 3$ has the intercept (0,3).

This is because: $0^2 - 4 \times 0 + 3 = 3$.

x-intercept

The x -intercepts of all graphs are found by substituting $y = 0$ into the function rule.

This usually involves solving an equation.

If you are given the function rule in factorised form, the intercepts can be found using the 'null-factor rule'.

The x -intercepts of $y = (x+1)(x-3)$ are found by solving:

$$(x+1)(x-3) = 0.$$

The results of a multiplication are only ever zero if one or other of the quantities being multiplied are zero.

Thus either: $x+1=0 \Rightarrow x=-1$

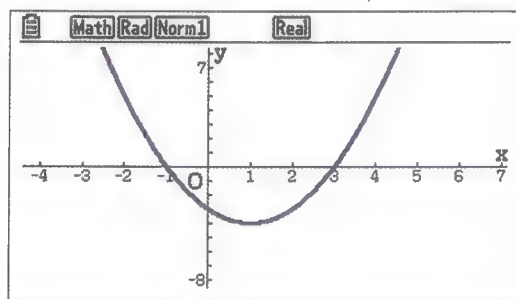
or: $x-3=0 \Rightarrow x=3$.

This gives the complete set of intercepts as:

$(-1,0)$, $(3,0)$ & $(0,-3)$.

The last is the intercept that results from $x = 0$. It also gives the line of symmetry (midway between the x -intercepts as the line $x = 1$).

The result can be confirmed using a graphic calculator.



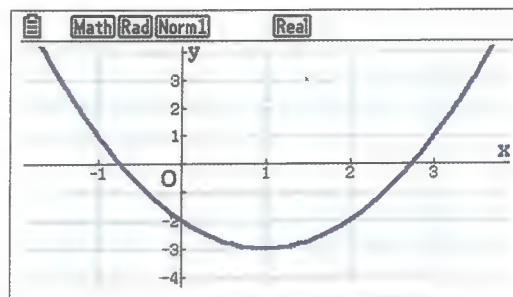
Beware

Beware of misusing the null-factor rule.

It is a very common mistake to look at a parabola such as:

$y = (x+1)(x-3)+1$ and conclude that it also has intercepts at $(-1,0)$ & $(3,0)$. This is **not** true.

However (think translations here), we can conclude that the graph passes through $(-1,1)$ & $(3,1)$ and also that the line of symmetry is $x = 1$.



Quadratic formula

If the rule is not able to be factorised, the quadratic formula can be used to find the x -intercepts.

Example B.3.5

Sketch the graph of $y = x^2 - x - 1$.

The y -intercept is $(0,-1)$.

The x -intercept is found by solving $x^2 - x - 1 = 0$. This expression will not readily factorise and so we need to use the quadratic formula:

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Never forget to check that the quadratic is being equated to zero (it is).

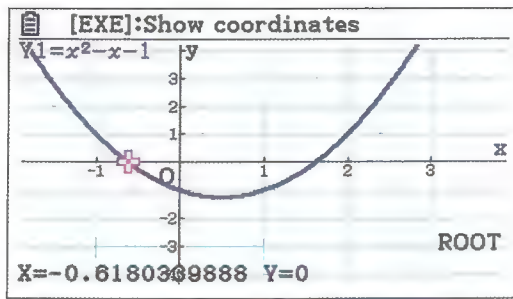
Next, list the three parameters: $a = 1$, $b = -1$ & $c = -1$.

Then substitute into the formula:

$$\begin{aligned} x &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times (-1)}}{2 \times 1} \\ &= \frac{1 \pm \sqrt{1+4}}{2} \\ &= \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2} \end{aligned}$$

The intercepts are: $\left(\frac{1+\sqrt{5}}{2}, 0\right)$ and $\left(\frac{1-\sqrt{5}}{2}, 0\right)$.

These are approximately $(1.62, 0)$ & $(-0.62, 0)$.



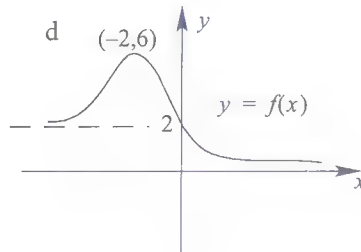
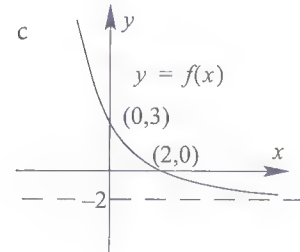
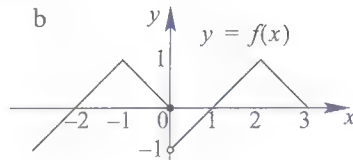
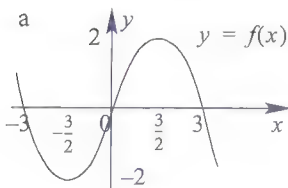
Exercise B.3.3

1. Sketch the graphs of:

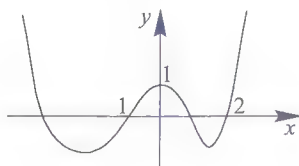
i $y = f(-x)$

ii $y = -f(x)$

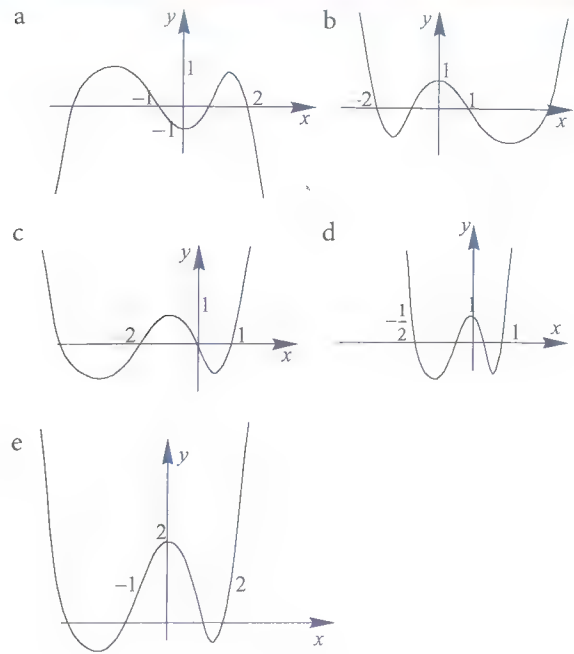
for each of the following.



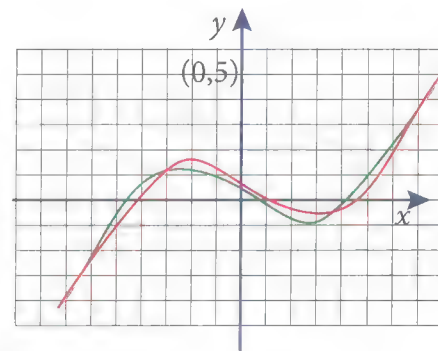
2. The diagram below shows the graph of the function $y = f(x)$.



Find the equation in terms of $f(x)$ for each of the following graphs.

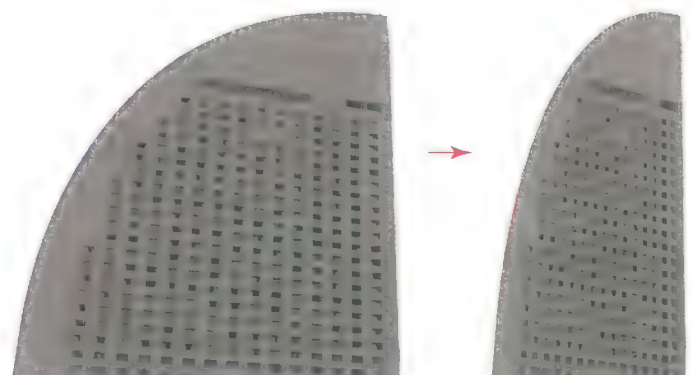


3. The function f has the graph shown:



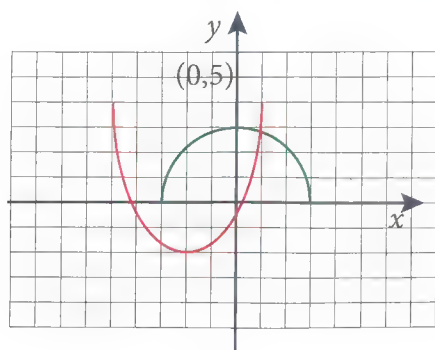
Sketch the graph of $y = -f(-x)$.

4. Describe the transformation implied by this diagram:



5. A weather graph shows how the temperature ($^{\circ}\text{C}$) varies throughout the day. What transformation must be used to convert this to a Fahrenheit graph?

5. Describe the transformation that changes the green graph into the red graph



6. During an experiment, the following data was collected using an automatic data collection device.

Time	1	2	3	4	5	6
Voltage	233	215	228	219	222	224

Time	7	8	9	10	11	12
Voltage	233	216	216	227	229	215

Subsequently, it was discovered that the equipment was making systematic errors. This means that every reading is incorrect in the same way.

The adjusted readings are:

Time	1	2	3	4	5	6
Voltage	251.3	231.5	245.8	235.9	239.2	241.4

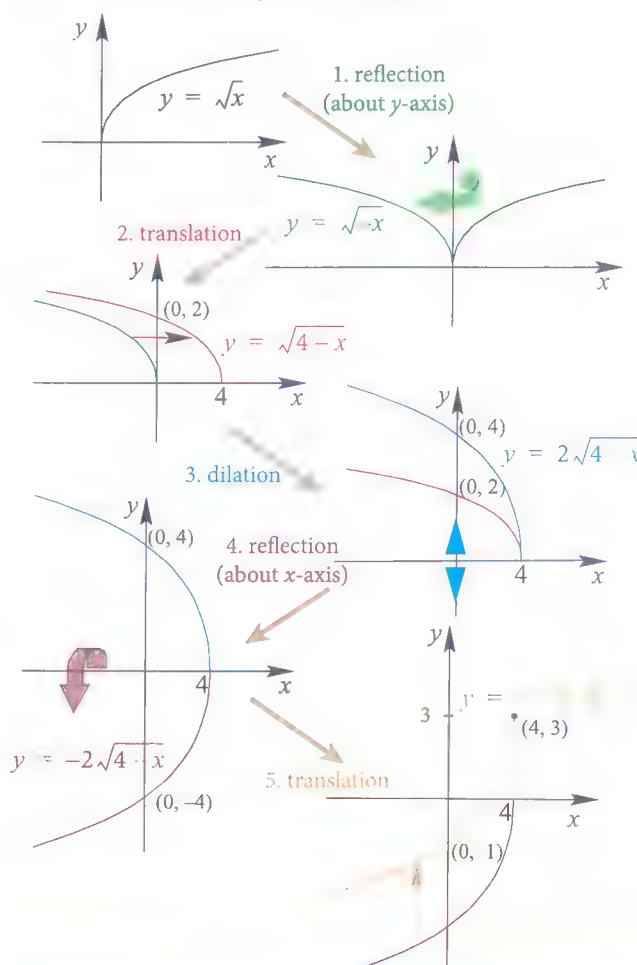
Time	7	8	9	10	11	12
Voltage	251.3	232.6	232.6	244.7	246.9	231.5

Given that the correction that was made is linear, what was it?

We start by considering the function $f(x) = \sqrt{x}$, then, the expression $y = 3 - 2\sqrt{4-x}$ can be written in terms of $f(x)$ as follows: $y = 3 - 2f(4-x)$ or $y = -2f(4-x) + 3$. This represents:

1. reflection about the y -axis (due to the ' $-x$ ' term)
2. translation of 4 units to the right (due to the ' $4-x$ ' term). Note: $4-x = -(x-4)$
3. dilation of factor 2 along the y -axis (due to the $2f(x)$ term)
4. reflection about the x -axis (due to the ' $-$ ' in front of the $2f(x)$ term)
5. translation of 3 units up (due to the '+3' term)

We produce the final graph in stages:



Compositions

We now consider a combination of the transformations we have looked at so far.

Example B.3.6

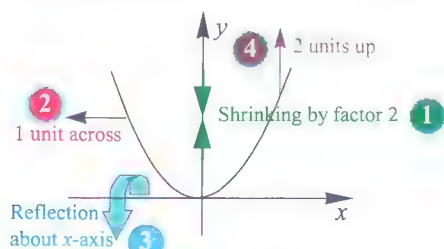
Sketch the graph of $y = 3 - 2\sqrt{4-x}$.

The previous example gave a step-by-step account of how to produce the final graph, however, there is no need to draw that many graphs to produce the final outcome. We can reduce the amount of work involved by including all the transformations on one set of axes and then produce the final graph on a new set of axes.

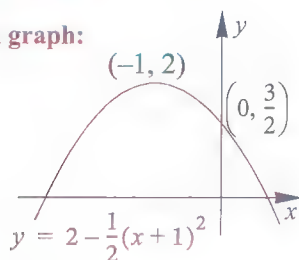
Example B.3.7

Sketch the graph of $y = 2 - \frac{1}{2}(x+1)^2$.

If we consider the function $f(x) = x^2$, the graph of $y = 2 - \frac{1}{2}(x+1)^2$ can be written as $y = 2 - \frac{1}{2}f(x+1)$. This represents a 'shrinking' effect of factor 2 from the x -axis followed by a reflection about the x -axis then a translation of 1 to the left and finally a translation of 2 units up.



Final graph:



At this stage we have not looked at the x - or y -intercepts, although these should always be determined.

Important note!

Note the order in which we have carried out the transformations - although there is some freedom in this - there are some transformations that must be carried out before others.

You should try to alter the order in which the transformations in Example B.3.7 have been carried out. For example, does it matter if we apply Step 2 before Step 1? Can Step 2 be carried out after Step 4?

b $f(x) = 4 - \frac{1}{2}x^2$

c $f(x) = 1 - \frac{1}{8}x^3$

d $f(x) = -(x+2)^2 + 3$

e $f(x) = \frac{2}{1-x}$

f $f(x) = 1 - \frac{1}{x+2}$

g $f(x) = -(x-2)^3 - 2$

h $f(x) = \frac{1}{2(2-x)}$

i $f(x) = 4 - \frac{2}{x^2}$

j $f(x) = 3 - \frac{2}{1-x}$

k $f(x) = -(1-x)^2$

l $f(x) = 4\sqrt{9-x}$

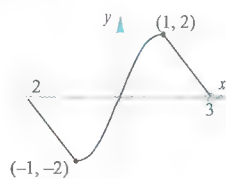
m $f(x) = \frac{2}{(2x-1)^2}$

Exercise B.3.4

1. Sketch the graphs of the following.

a $f(x) = -3x^2 + 9$

2. The graph of $y = f(x)$ is shown.



Use it to sketch the graphs of:

a $y = f(x - 1)$

b $y = f(x) - 1$

c $y = f(x + 1)$

d $y = 1 - f(x)$

e $y = 1 + f(-x)$

f $y = 2 - f(-x)$

g $y = -\frac{1}{2}f(x)$

h $y = f(-2x)$

Answers



B.4 Special Functions

SL 2.6

SL 2.7

SL 2.8

SL 2.9

SL 2.6 The Quadratic Function

A **quadratic function** has the general form:

$$f(x) = ax^2 + bx + c, a \neq 0 \text{ and } a, b, c \in \mathbb{R}$$

All quadratic functions have **parabolic graphs** and have a **vertical axis of symmetry**.

If $a > 0$, the parabola is concave up:



If $a < 0$ the parabola is concave down:



General properties of parabolic graphs

1. **y-intercept:**

This occurs when $x = 0$, so that:

$$y = f(0) = a \times (0)^2 + b \times (0) + c = c$$

That is, the curve passes through the point $(0, c)$.

2. **x-intercept(s):**

This occurs where $f(x) = 0$.

Therefore we need to solve $ax^2 + bx + c = 0$.

To solve we either factorise and solve or use the quadratic formula, which would provide the solution(s):

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3. **Axis of symmetry:**

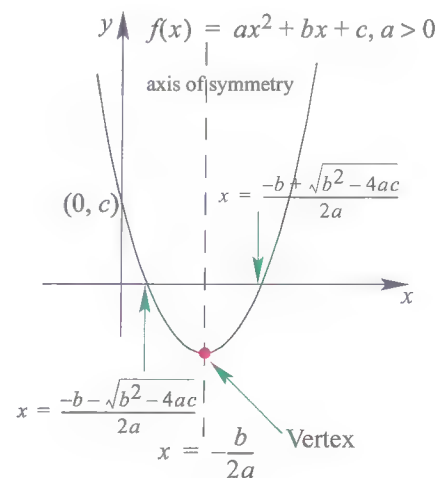
This occurs at $x = -\frac{b}{2a}$.

4. **Vertex (turning point):**

The vertex occurs when $x = -\frac{b}{2a}$.

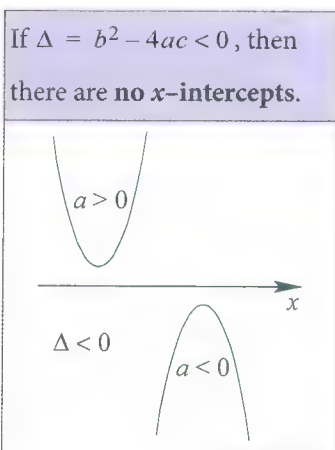
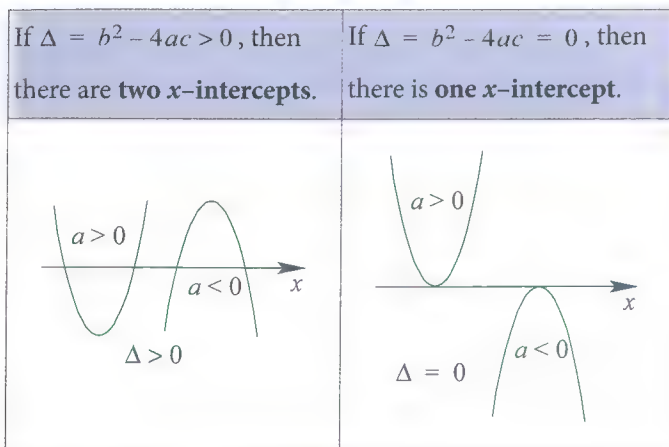
Then, to find the y-value, find $f\left(-\frac{b}{2a}\right)$

In summary:



Geometrical Interpretation and the Discriminant

Just as it was the case for linear functions, solving the quadratic $ax^2 + bx + c = k$ is geometrically equivalent to finding where the parabola with function $f(x) = ax^2 + bx + c$ meets the graph (horizontal straight line) $y = k$. Then, when $k = 0$, we are finding where the parabola meets the line $y = 0$, i.e. we are finding the x -intercept(s). Based on our results of the discriminant about the number of solutions to the equation $ax^2 + bx + c = 0$, we can extend these results to the following:



Sketching the graph of a quadratic function

Two methods for sketching the graph of $f(x) = ax^2 + bx + c, a \neq 0$ are:

Method 1: The intercept method

i.e. expressing $f(x) = ax^2 + bx + c, a \neq 0$ in the form $f(x) = a(x-p)(x-q)$

This involves:

Step 1

Finding the x -intercepts (by solving $ax^2 + bx + c = 0$)

Step 2

Finding the y -intercept (finding $f(0)$)

Step 3

Sketching the parabola passing through the three points.

Method 2: The turning-point form

i.e. expressing $f(x) = ax^2 + bx + c, a \neq 0$ in the form:

$$f(x) = a(x-h)^2 + k$$

This involves:

Step 1

Expressing $f(x) = ax^2 + bx + c, a \neq 0$ in the form $a(x-h)^2 + k$ (by completing the square)

Step 2

Using the turning point (h, k)

Step 3

Finding the y -intercept (finding $f(0)$)

Step 4

Sketching the parabola passing through the two points.

Example B.4.1

Sketch the graphs of:

a $f(x) = x^2 - 6x + 8$

b $f(x) = 3x^2 + 12x + 4$

using:

i the intercept method.

ii the turning point method.

- a For the intercept method we start by factorizing the quadratic:

$$i \quad f(x) = x^2 - 6x + 8 = (x-4)(x-2).$$

Then, for the x -intercepts we have:

$$f(x) = 0 \Leftrightarrow (x-4)(x-2) = 0 \Leftrightarrow x = 2 \text{ or } x = 4$$

Next, the y -intercept:

$$x = 0 \Rightarrow f(0) = 0^2 - 6(0) + 8 = 8.$$

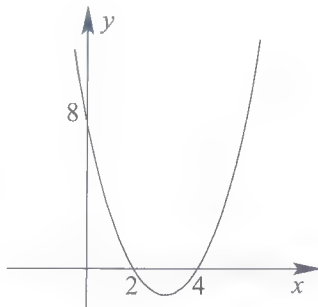
Summary:

Parabola passes through (2, 0), (4, 0) and (0, 8).

Note, the axis of symmetry is the midpoint of the x -intercepts,

$$\text{i.e. } x = \frac{2+4}{2} = \frac{6}{2} = 3.$$

$$\text{Or use } x = -\frac{b}{2a} = -\frac{-6}{2} = 3$$



- ii We first complete the square:

$$f(x) = x^2 - 6x + 8 = (x^2 - 6x + 9) - 1 = (x-3)^2 - 1$$

Therefore, the parabola has a turning point at (3, -1). Review the explanations of translation if this is not clear to you.

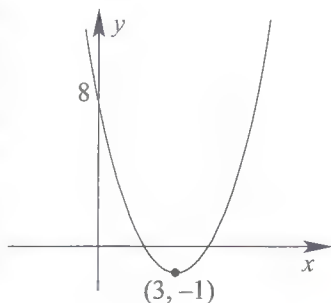
Then find the y -intercept:

$$x = 0 \Rightarrow f(0) = 0^2 - 6(0) + 8 = 8$$

Summary:

Parabola passes through (3, -1) and (0, 8).

Notice that both graphs are the same but display different information. Each display has its strengths and weaknesses – it depends on what information is important for the question. Of course, including all the information will rid us of any problems.



- b For the intercept method we start by factorizing the quadratic (completing the square):

$$f(x) = 3x^2 + 12x + 4 = 3(x^2 + 4x + 4) - 12 + 4$$

$$= 3(x+2)^2 - 8 \text{ (next use difference of two squares).}$$

$$= [\sqrt{3}(x+2) - 2\sqrt{2}][\sqrt{3}(x+2) + 2\sqrt{2}].$$

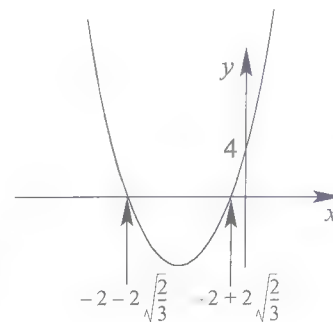
Then, for the x -intercepts we have:

$$f(x) = 0 \Rightarrow [\sqrt{3}(x+2) - 2\sqrt{2}][\sqrt{3}(x+2) + 2\sqrt{2}] = 0$$

$$\therefore x = -2 + 2\sqrt{\frac{2}{3}} \text{ or}$$

$$x = -2 - 2\sqrt{\frac{2}{3}}$$

Note: It would have been quicker to find the intercepts if we had used the quadratic formula!



Next, the y -intercept:

$$x = 0 \Rightarrow f(0) = 3(0)^2 + 12(0) + 4 = 4.$$

Summary:

Parabola passes through $\left(-2 + 2\sqrt{\frac{2}{3}}, 0\right)$, $\left(-2 - 2\sqrt{\frac{2}{3}}, 0\right)$ and (0, 4).

- ii We first complete the square:

$$f(x) = 3x^2 + 12x + 4 = 3(x^2 + 4x + 4) - 12 + 4$$

(completing the square)

$$= 3(x+2)^2 - 8$$

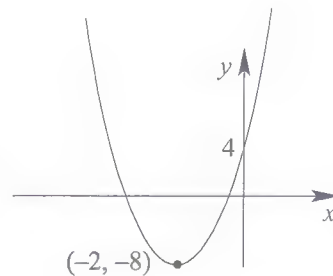
Therefore, the parabola has a turning point at (-2, -8)

Then find the y -intercept:

$$x = 0 \Rightarrow f(0) = 3(0)^2 + 12(0) = 4$$

Summary:

Parabola passes through (-2, -8) and (0, 4).



Example B.4.2

For what value(s) of k will the function: $f(x) = x^2 + 6x + k$

- a cut the x -axis twice?
- b touch the x -axis?
- c have no x -intercepts?

We will need to use the relationship between the number of x -intercepts of the function $f(x) = x^2 + 6x + k$ and the number of solutions to the equation $x^2 + 6x + k = 0$. To do this we first find the discriminant: $\Delta = b^2 - 4ac = (6)^2 - 4 \times 1 \times k = 36 - 4k$.

If the function cuts twice, then $\Delta > 0 \Rightarrow 36 - 4k > 0 \Leftrightarrow k < 9$

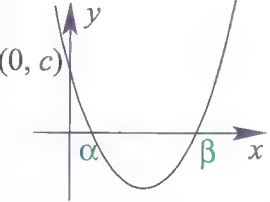
If the function touches the x -axis, then we have repeated solutions (or roots). In this case we have only one solution, so $\Delta = 0 \Rightarrow 36 - 4k = 0 \Leftrightarrow k = 9$.

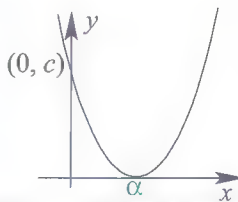
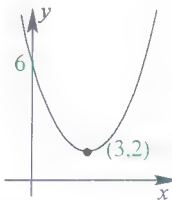
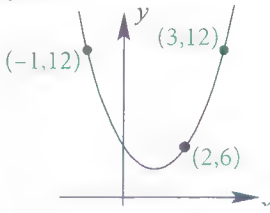
If the function has no x -intercepts, then $\Delta < 0 \Rightarrow 36 - 4k < 0 \Leftrightarrow k > 9$.

So far we have sketched a graph from a given function, but what about finding the equation of a given graph?

Finding the equation from a graph

If sufficient information is provided on a graph, then it is possible to obtain the equation that corresponds to that graph. When dealing with quadratics there are some standard approaches that can be used (depending on the information provided).

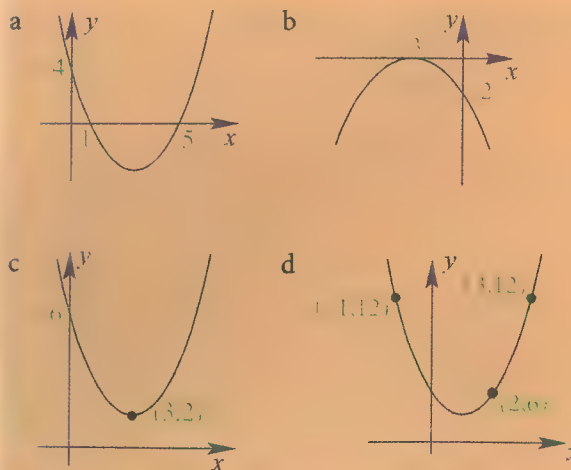
Information provided	Process
Graph cuts the x -axis at two points: 	Use the function $f(x) = k(x - \alpha)(x - \beta)$ and then use the point $(0, c)$ to solve for k .

Information provided	Process
Graph touches the x -axis at $x = \alpha$: 	Use the function: $f(x) = k(x - \alpha)^2$ and then use the point $(0, c)$ to solve for k .
Graph does not meet the x -axis: 	Use the function $f(x) = k(x - \alpha)^2 + \beta$ and then use the point $(0, c)$ to solve for k .
Three arbitrary points are given: 	Use the function: $f(x) = ax^2 + bx + c$ and then set up and solve the system of simultaneous equations by substituting each coordinate into the function: $ax_1^2 + bx_1 + c = y_1$ $ax_2^2 + bx_2 + c = y_2$ $ax_3^2 + bx_3 + c = y_3$

Note: the process is identical for a downward concave parabola.

Example B.4.3

Find the equation defining each of the following graphs:



- a Using the form $f(x) = k(x - \alpha)(x - \beta)$ we have
 $f(x) = k(x - 1)(x - 5)$.

Next, when $x = 0, y = 4$, therefore,

$$4 = k(0 - 1)(0 - 5) \Leftrightarrow 4 = 5k \Leftrightarrow k = \frac{4}{5}.$$

Therefore, $f(x) = \frac{4}{5}(x - 1)(x - 5)$.

- b Graph touches x -axis at $x = -3$, therefore use
 $f(x) = k(x + 3)^2$.

As graph passes through $(0, -2)$, we have:

$$-2 = k(0 + 3)^2 \Leftrightarrow -2 = 9k \Leftrightarrow k = -\frac{2}{9}.$$

Therefore, $f(x) = -\frac{2}{9}(x + 3)^2$.

- c Graph shows turning point and another point, so use
the form $f(x) = k(x - \alpha)^2 + \beta$.

So we have, $f(x) = k(x - 3)^2 + 2$.

Then, as graph passes through $(0, 6)$, we have
 $6 = k(0 - 3)^2 + 2 \Leftrightarrow 4 = 9k \Leftrightarrow k = \frac{4}{9}$.

Therefore, $f(x) = \frac{4}{9}(x - 3)^2 + 2$.

- d As we are given three arbitrary points, we use the
general equation $f(x) = ax^2 + bx + c$.

From $(-1, 12)$ we have $12 = a(-1)^2 + b(-1) + c$

$$\text{i.e. } 12 = a - b + c - (1)$$

From $(2, 6)$ we have $6 = a(2)^2 + b(2) + c$

$$\text{i.e. } 6 = 4a + 2b + c - (2)$$

From $(3, 12)$ we have $12 = a(3)^2 + b(3) + c$

$$\text{i.e. } 12 = 9a + 3b + c - (3)$$

Solving for a, b and c we have:

$$(2) - (1): \quad -6 = 3a + 3b$$

$$\text{i.e. } -2 = a + b - (3)$$

$$(3) - (2): \quad 6 = 5a + b - (4)$$

$$(4) - (3): \quad 8 = 4a \Leftrightarrow a = 2$$

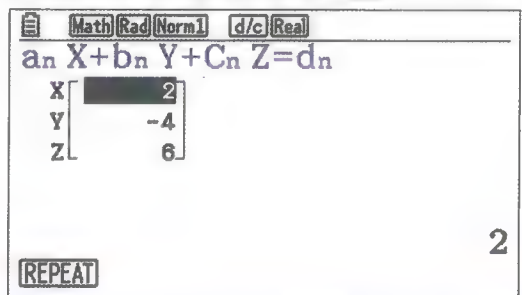
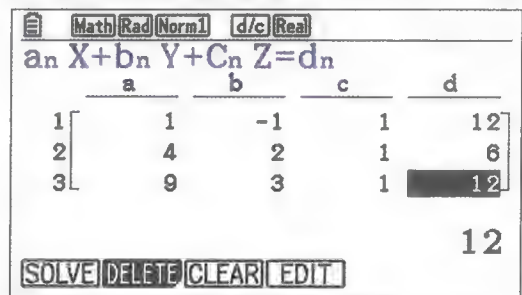
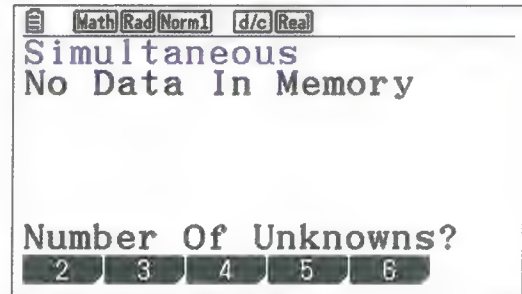
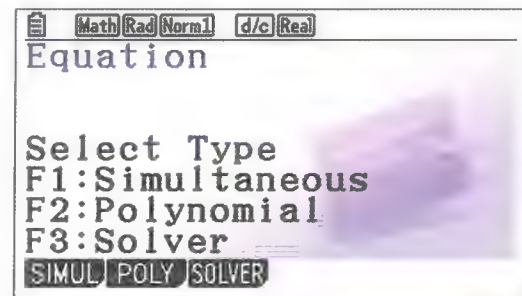
Substitute $a = 2$ into (3): $-2 = 2 + b \Leftrightarrow b = -4$.

Substitute results into (1):

$$12 = 2 - (-4) + c \Leftrightarrow c = 6.$$

Therefore function is $f(x) = 2x^2 - 4x + 6$.

Students should be aware that a technological solution is available using the equation solving modules available on most calculators.



Exercise B.4.1

1. Express the following functions in turning point form and hence sketch their graphs:

a $y = x^2 - 2x + 1$ b $y = x^2 + 4x + 2$

c $y = x^2 - 4x + 2$ d $y = x^2 + x - 1$

e $y = x^2 - x - 2$ f $y = x^2 + 3x + 1$

g $y = -x^2 + 2x + 1$

h $y = -x^2 - 2x + 2$

i $y = 2x^2 - 2x - 1$

j $y = -\frac{1}{2}x^2 + 3x - 2$

k $y = -\frac{x^2}{3} + x - 2$

l $y = 3x^2 - 2x + 1$

2. Find the axial intercepts of these quadratic functions (correct to 2 decimal places) and hence sketch their graphs:

a $y = x^2 + 3x + 2$ b $y = x^2 - x - 6$

c $y = 2x^2 - 5x - 3$ d $y = x^2 - 4$

e $y = x^2 + x - 5$ f $y = -x^2 + x + 6$

g $y = -x^2 + x + 1$

h $y = -2x^2 - 3x + 5$

i $y = 2x^2 + 5x - 3$

j $y = \frac{x^2}{3} - 2x + 3$

k $y = -\frac{x^2}{2} + x + 4$

l $y = 3x^2 - 2x - 4$

3. For the quadratic function $f(x) = 2x^2 - 4x + 1$, find:

a the equation of the axis of symmetry.

b the coordinates of the vertex.

c i x -intercept(s). ii y -intercept.

Hence, sketch the graph of the function.

4. For the quadratic function $f(x) = 7 + 4x - 2x^2$, find:

a the equation of the axis of symmetry.

b the coordinates of the vertex.

c i x -intercept(s). ii y -intercept.

Hence, sketch the graph of the function.

5. For what value(s) of k will the graph of:

$y = x^2 - 3x + k$

a touch the x -axis?

b cut the x -axis?

c never meet the x -axis?

6. For what value(s) of k will the graph of:

$y = kx^2 + 5x + 2$:

a touch the x -axis?

b cut the x -axis?

c never meet the x -axis?

7. For what value(s) of k will the graph of:

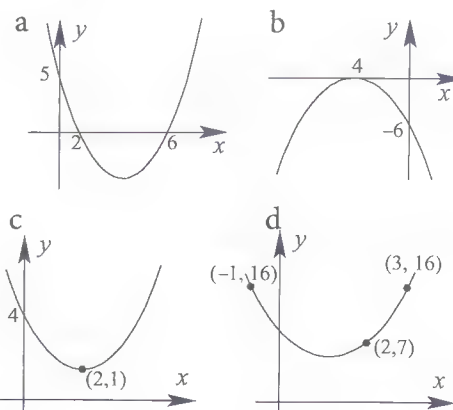
$y = kx^2 - 2x + k$:

a touch the x -axis?

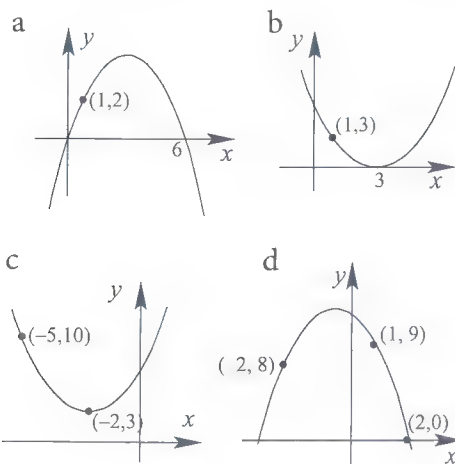
b cut the x -axis?

c never meet the x -axis?

8. Find the equation of the quadratic function with graph:



9. Find the equation of the quadratic function with graph:



Modelling using parabolas

Modelling is the process of taking a physical object or phenomenon and producing a mathematical object (function, equation etc.) that behaves in the same way.

There are many reasons for doing this amongst which are:

aiding in the design of buildings, machines etc.; predicting the future; explaining processes such as chemical reactions; navigating; managing money; forming business strategies.

The exact process of constructing, evaluating and using a model varies from application to application.

Reaction Yields

Some chemical reactions only proceed under pressure.

The 'yield' of such a reaction is the amount of a desired product that is generated.

In commercial situations, it is desirable to design such processes to maximise yield.

A new process is being studied with a view to designing a production plant.

The results of laboratory experiments are:

Pressure	Yield
0.3	37.4
0.5	42.9
0.7	47.7
0.9	51.8
1.1	55.2
1.3	58.0
1.5	60.1

Pressure (P) is measured in Mega-Pascals and the yield (Y) is given in %.

If this is modelled by a quadratic of the form:

$Y = aP^2 + bP + c$ where a , b & c are constants, what is the optimum pressure that should be designed into the production plant?

The constants can be found by selecting three data points. These should be spread through the data (ie. beginning, middle and end).

If we choose: (0.3, 37.4), (0.9, 51.8) & (1.5, 60.1), we get three simultaneous equations in the three unknowns.

$$0.3^2 a + 0.3b + c = 37.4$$

$$0.9^2 a + 0.9b + c = 51.8$$

$$1.5^2 a + 1.5b + c = 60.1$$

This can be solved by 'hand' (using elimination), but a calculator can save time.

If using TI models choose (3) Algebra, (7 & 1) Solve system of equations.

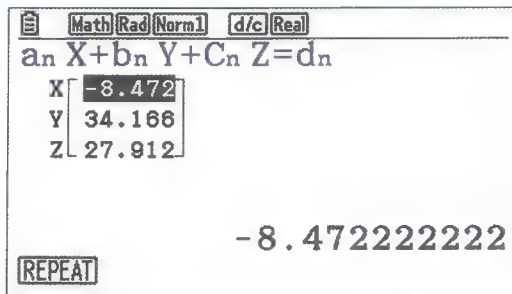
If using a Casio, choose the equation mode. (F1) Simultaneous.

In both cases, specify a 3 by 3 system.

Enter the coefficients of the equations.

Math Rad Norm1 d/c Real				
$a_n X + b_n Y + c_n Z = d_n$				
	a	b	c	d
1	0.09	0.3	1	37.4
2	0.81	0.9	1	51.8
3	2.25	1.5	1	60.1
				60.1
SOLVE DELETE CLEAR EDIT				

Choose F1, Solve.



Note that these values are not X, Y & Z , but a, b & c .

The modelling equation is: $Y = -8.47P^2 + 34.2P + 27.9$.

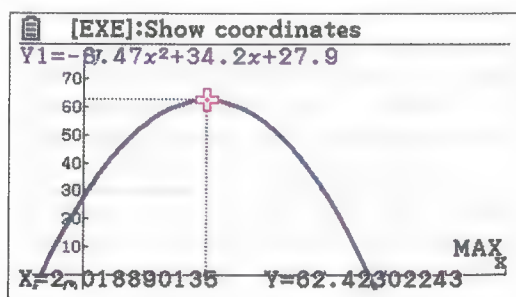
The set of data, the model predictions and the differences between the two are summarised in this table:

Pressure	Yield	Model	Difference
0.3	37.4	37.40	0.002
0.5	42.9	42.88	0.017
0.7	47.7	47.69	0.010
0.9	51.8	51.82	-0.019
1.1	55.2	55.27	-0.071
1.3	58	58.05	-0.046
1.5	60.1	60.14	-0.043

This suggests that the model has matched the data quite well with some of the predictions being too big and some too small (the differences are both positive and negative and all are quite small).

The value of particular interest is the maximum point of this modelling curve as this will result in the commercial process operating optimally.

Using a Casio graphic calculator and G-Solv to find the maximum point:



This suggests that the yield will be maximised if the pressure is set to 2.0 Mega Pascals with a yield of 62%.

Remember that models may deviate from the truth!

Exercise B.4.2

1. Use the intercepts method to sketch:

a $y = (x-1)(x+2)$

b $y = (x+1)(x-3)$

c $y = (1-x)(x+2)$

d $y = x(x-1)$

e $y = x^2 - 1$

f $y = x^2 - x - 2$

2. Use transformations to sketch:

a $y = (x-1)^2 - 1$

b $y = (x+2)^2 - 3$

c $y = -(x+2)^2 + 3$

d $y = -(x-1)^2 + 4$

e $y = 2(x-1)^2 - 3$

f $y = -2(x-1)^2 + 3$

3. Sketch:

a $y = x^2 - 2x + 1$

b $y = x^2 + 2x + 1$

c $y = x^2 - 4$

d $y = 2x^2 - x - 3$

e $y = 2x^2 + 3x - 2$

f $y = 6x^2 + x - 1$

4. Find the equation of the parabola with y -intercept (0,4) and the single x -intercept (1,0).
5. A parabola has a maximum point at (2,1) and an x -intercept at (3,0). Find its rule.
6. Sketch the parabola $y = 2x^2 - 5x - 3$ and hence solve the inequation $2x^2 - 5x - 3 > 0$.
7. A company estimates that its income from sales is modelled by the function:

$$I(t) = -t^2 + 10t + 1$$

The operating costs are modelled by $C(t) = 0.2t + 5$

where I & C are measured in 1,000s \$ and t in years.

For what period can the company expect to be profitable?

8. A rescue line is fired from the shoreline to a ship in distress off the coast. The line is attached to a projectile which must pass over the deck of the ship and land in the sea beyond so that the line lands on the deck. The trajectory of the projectile is modelled by:

$$h(x) = \frac{x}{2} - \frac{x^2}{10}$$

where h is the height and x the distance off shore measured in 100s of metres. The ship is 450 metres from the beach and the deck is 12 metres from the sea. Will the rescue line hit the deck?

SL 2.7 Quadratic Equations

As we have observed, there is a direct connection between sketching the graphs of quadratic functions and solving quadratic equations and inequations.

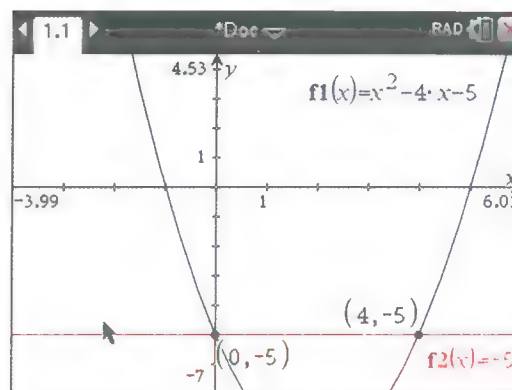
Example B.4.4

Use graphical methods to solve:

- a $x^2 - 4x - 5 = -5$
- b $x^2 - 4x - 5 > -5$
- c $x^2 - 4x - 5 = 1$
- d $x^2 - 4x - 5 \leq 1$

a We can solve this equation by sketching:

- i $f(x) = x^2 - 4x - 2$ and finding the intercepts with the horizontal line $y = -5$.



The solutions are $x = 0$ or 4 .

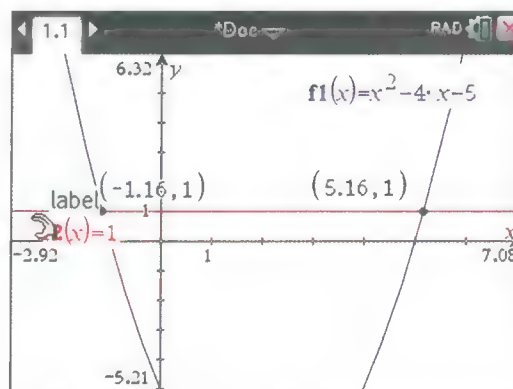
- ii Alternatively, rearrange the original equation to read $x^2 - 4x = 0$, sketching $f(x) = x^2 - 4x$ and find the x -intercepts. This is probably the easier method as $f(x) = x^2 - 4x = x(x - 4)$. The intercepts follow immediately as $x = 0$ or 4 .

- b We are looking at $x^2 - 4x - 5 > -5$. In the above calculator screen, we are looking for parts of the graph where $f1(x) = x^2 - 4x - 2$ (blue) is above $f2(x) = -5$ (red). Note the direction of the inequality sign.

The solution is: $x < 0$ or $x > 4$.

- c $x^2 - 4x - 5 = 1$

Modifying the above graphical analysis to $f2(x) = 1$ and finding the intercepts, gives the following.



This is where the shortcomings of graphical methods become apparent as we only get approximate solutions. In this case: $x = -1.16$ or 5.16 .

- d In this case we are looking for the blue graph to be below the red one: $-1.16 \leq x \leq 5.16$.

Example B.4.5

Use algebraic methods to solve:

a $x^2 - 4x - 5 = 1$

b $x^2 - 4x - 5 \leq 1$

- a Completing the square. This is most easily done with all the pronumerals on their own on one side of the equation. $x^2 - 4x - 5 = 1 \Rightarrow x^2 - 4x = 6$

$$x^2 - 4x = 6$$

$$x^2 - 4x + 4 = 10$$

Adding 4 to both sides is chosen as it makes the left hand side a complete square. The solution proceeds as follows:

$$x - 2 = \pm\sqrt{10}$$

$$x = 2 \pm \sqrt{10}$$

$$2 - \sqrt{10} \approx -1.162$$

$$2 + \sqrt{10} \approx 5.162$$

This confirms the approximate solution obtained graphically.

Alternatively, we can use the formula:

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

To use the formula, it is important to have all non-zero terms on one side and zero on the other.

$$x^2 - 4x - 5 = 1 \Rightarrow x^2 - 4x - 6 = 0.$$

Next, identify the parameters a , b & c . In this case, these are: $a = 1$, $b = -4$, $c = -6$.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \times 1 \times (-6)}}{2 \times 1} \\ &= \frac{4 \pm \sqrt{16 + 24}}{2} \\ &= \frac{4 \pm \sqrt{40}}{2} \\ &= \frac{4 \pm \sqrt{4 \times 10}}{2} \\ &= \frac{4 \pm 2\sqrt{10}}{2} \\ &= 2 \pm \sqrt{10} \end{aligned}$$

This is as obtained by completing the square. This is because the formula is obtained by completing the square (see below).

- b Solving an inequation is still best done by using a graph (as in the previous example) to identify the correct interval(s) to give $2 - \sqrt{10} \leq x \leq 2 + \sqrt{10}$

Optional Proof

Let: $ax^2 + bx + c = 0$ and complete the square:

$$ax^2 + bx + c = 0$$

$$ax^2 + bx = -c$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} = -\frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right) = \pm \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}$$

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}$$

$$= -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$= -\frac{b}{2a} \pm \frac{1}{2a} \sqrt{b^2 - 4ac}$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example B.4.6

Find exact solutions of:

a $3x^2 + 4x + 7 = 12$

b $3x^2 + 4x + 7 \leq 12$

a $3x^2 + 4x + 7 = 12$

Using the formula, we must first rearrange to get zero on one side: $3x^2 + 4x - 5 = 0$.

Next, identify a , b , c : $a = 3$, $b = 4$, $c = -5$.

Now use the formula:

$$\begin{aligned}
 x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
 &= \frac{-4 \pm \sqrt{4^2 - 4 \times 3 \times (-5)}}{2 \times 3} \\
 &= \frac{-4 \pm \sqrt{16 + 60}}{2 \times 3} \\
 &= \frac{-4 \pm \sqrt{76}}{6} \\
 &= \frac{-4 \pm \sqrt{4 \times 19}}{6} \\
 &= \frac{-4 \pm 2\sqrt{19}}{6} \\
 &= \frac{-2 \pm \sqrt{19}}{3}
 \end{aligned}$$

- b The parabola relating to this problem is 'vertex down' so the solution to $3x^2 + 4x + 7 \leq 12$ is:

$$\frac{-2 - \sqrt{19}}{3} \leq x \leq \frac{-2 + \sqrt{19}}{3}$$

Exercise B.4.3

1. By using a factorisation process, solve for the given variable.

a $x^2 + 10x + 25 = 0$

b $x^2 - 10x + 24 = 0$

c $3x^2 + 9x = 0$

d $x^2 - 4x + 3 = 0$

e $(3 - u)(u + 6) = 0$

f $3x^2 + x - 10 = 0$

g $3v^2 - 12v + 12 = 0$

h $y(y - 3) = 18$

i $(x + 3)(x + 2) = 12$

j $(2a - 1)(a - 1) = 1$

2. Without using the quadratic formula, solve for the given variable.

a $u + \frac{1}{u} = -2$

b $x + 2 = \frac{35}{x}$

c $5x - 13 = \frac{6}{x}$

d $\frac{x}{2} - \frac{1}{x+1} = 0$

e $y + 1 = \frac{4}{y+1}$

f $v + \frac{20}{v} = 9$

3. By completing the square, solve for the given variable.

a $x^2 + 2x = 5$

b $x^2 + 4 = 6x$

c $x^2 - 2x = 4$

d $4x^2 + x = 2$

e $2y^2 = 9y - 1$

f $3a^2 - a = 7$

4. Use the quadratic formula to solve these equations.

a $x^2 - 3x - 7 = 0$ b $x^2 - 5x = 2$

c $x^2 - 3x - 6 = 0$ d $x^2 = 7x + 2$

e $x(x + 7) = 4$ f $x^2 + 2x - 8 = 0$

g $x^2 + 2x - 7 = 0$ h $x^2 + 5x - 7 = 0$

i $x^2 - 3x - 7 = 0$ j $x^2 - 3x + 9 = 0$

k $x^2 + 9 = 8x$ l $4x^2 - 8x + 9 = 0$

m $4x^2 = 8x + 9$ n $5x^2 - 6x - 7 = 0$

o $5x^2 - 12x + 1 = 0$

p $7x^2 - 12x + 1 = 0$

5. For what value(s) of p does the equation $x^2 + px + 1 = 0$ have:

a no real solutions?

b one real solution?

c two real solutions?

6. Find the values of m for which the quadratic $x^2 + 2x + m = 0$ has:
 - a no real solutions?
 - b one real solution?
 - c two real solutions?
7. Find the values of m for which the quadratic $x^2 + mx + 2 = 0$ has:
 - a no real solutions?
 - b one real solution?
 - c two real solutions?
8. Find the values of k for which the quadratic $2x^2 + kx + 9 = 0$ has:
 - a no real solutions?
 - b one real solution?
 - c two real solutions?
9. Consider the equation $x^2 + 2x = 7$. Prove that this equation has two real roots.
10. Find the value(s) of p such that the equation $px^2 - px + 1 = 0$ has exactly one real root.
11. Prove that the equation $kx^2 + 3x = k$ has two real solutions for all non-zero real values of k .
12. Solve the following inequalities:
 - i $x^2 - 4x + 3 \leq 0$
 - ii $x^2 - x \geq 2$
 - iii $x^2 + 2x < 4$
 - iv $3 - x^2 \geq 7x$
 - v $x(x + 2) \leq -3$
 - vi $x^2 + 1 \leq -2x$
 - vii $3x^2 + 1 \leq 7x$
 - viii $1 - x^2 \geq 3x$
 - ix $\frac{x^2}{4} + \frac{1}{2} \leq \frac{x}{3}$

$$x \quad \frac{1}{x+1} + x \leq 2$$

13. The pressure (P atmospheres) in a chemical reaction at time t (hours) is modelled by $P(t) = 11 + 10t - t^2$ where $0 \leq t \leq 10$. For what percentage of the time is the pressure greater than 27 atmospheres?
14. A parabolic ceremonial arch is fixed to a wall 6 m from the ground. The highest point is 14 m (horizontally) from the wall and is 25.6 m from the ground. The other end of the arch is supported by a 15.6 m pillar that is 24 m from the wall.
 - a Find a function that models the shape of the arch.
 - b Find the width of the part of the arch that is more than 20 m above the ground.

Theory of Knowledge

It is a common misunderstanding to think that mathematics is perfect. Every question has a right answer and being a student of mathematics involves knowing as many of these right answers as possible. This is far from the truth.

Consider this argument:

Let $x = 1$.

It follows that $x^2 - 1 = x - 1$ since both are zero.

The left hand side will factorise using the difference of two squares.

$$(x + 1)(x - 1) = x - 1$$

Next, divide both sides by $x - 1$ to get:

$$x + 1 = 1$$

But, remember that we started with 'Let $x = 1$ ', so it follows that:

$$2 = 1$$

What has gone wrong?

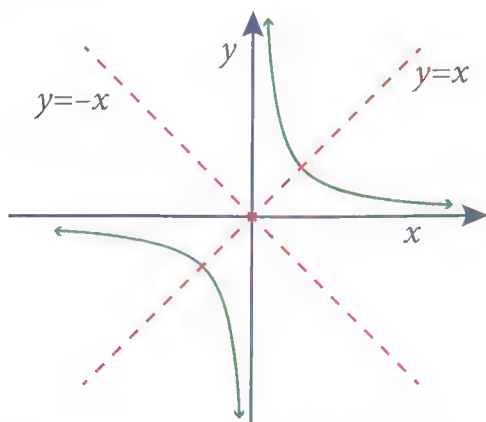
The false step is 'divide both sides by $x - 1$ ' as this quantity is zero. We can conclude that division by zero must be a banned process. Are there any other 'banned processes' in mathematics that you can think of?

SL 2.8 The Reciprocal Function

The basic reciprocal function is: $y = \frac{1}{x}, x \neq 0$.

The exclusion of zero from the domain is as a result of the ban on division by zero that we have just discussed.

Graphically, this is:



The curve is symmetric about the two red dotted red lines. It is also rotationally symmetric about the origin.

Note also that the symmetry about $y = x$ means that the basic reciprocal function is its own inverse.

This is a statement of the fact that the reciprocal of the reciprocal of a number is equal to the number (zero excepted).

$$\frac{1}{\left(\frac{1}{x}\right)} = x, x \neq 0$$

All the principles of transformations apply to the reciprocal function.

Example B.4.7

Sketch the graphs of:

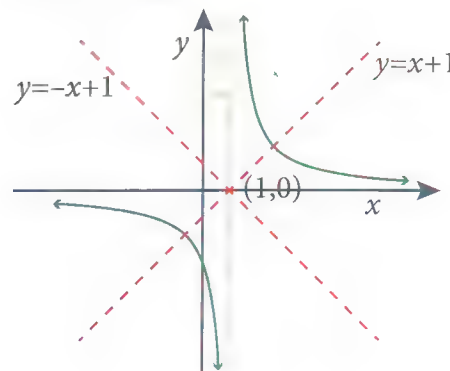
i $y = \frac{1}{x-1}, x \neq 1$

ii $y = \frac{1}{x+2} - 1, x \neq -2$

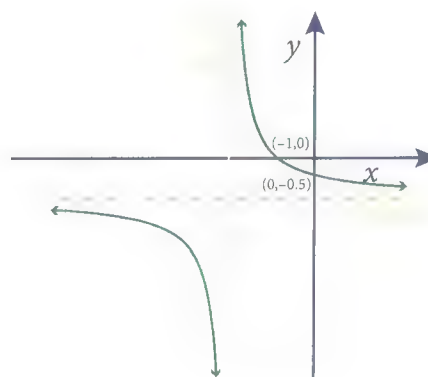
iii $y = \frac{-1}{x-1} + 1, x \neq 1$

iv $y = \frac{-1}{2x+1} - 2, x \neq -\frac{1}{2}$

- i This is a translation of 1 unit to the right. The asymptotes and lines of symmetry are also translated 1 unit to the right.



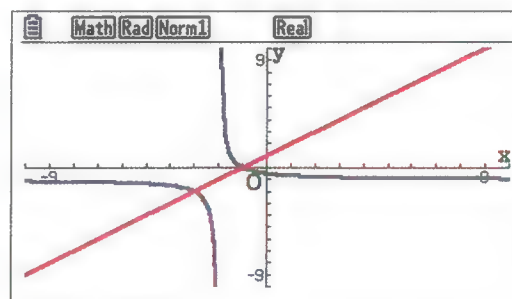
- ii This is a translation of 2 units to the left and 1 unit down.



If using a graphic calculator, there are several pitfalls to be aware of.

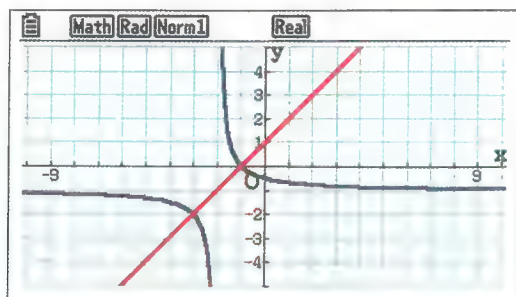
The first is that the calculator may join the two arms of the hyperbola (at least on some zoom settings). This should not be reflected in any sketches.

Another is that, since most calculator screens are rectangular, commonly used graph windows do not show equal scales on the axes. This will distort the line of symmetry.

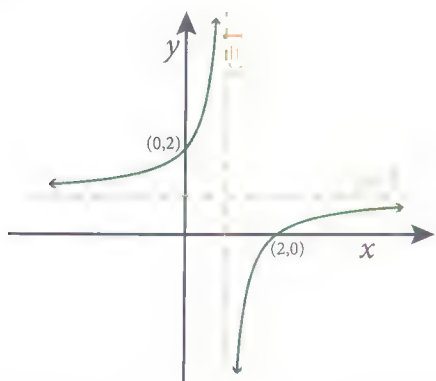


Note that the symmetry line does not appear to be at 45° to the axes.

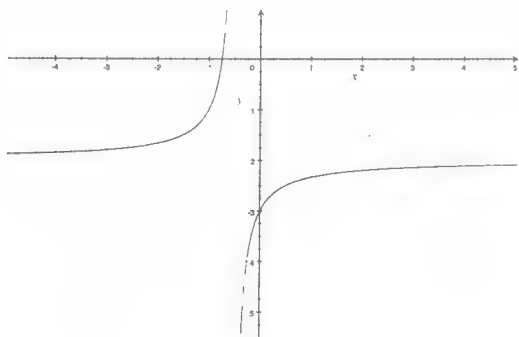
A better view is to choose the ZOOM / SQUARE options that are offered by many calculators. This shows the symmetry more clearly.



- iii This involves a left/right reflection (the minus sign in the numerator) as well as the translations 1 up and 1 right.



- iv This time we provide a graph from a graphing package:



Instead of thinking about transformations, look at the characteristics of the curve - as we have done for other graphs.

The function is: $y = \frac{-1}{2x+1} - 2, x \neq -\frac{1}{2}$

The vertical asymptote occurs when the denominator is zero. This happens when $x = -\frac{1}{2}$.

The horizontal asymptote occurs when x is very large. This makes the denominator large and its reciprocal small.

This leaves the constant part as the asymptote $y = -2$.

Set x to zero to find the y intercept: $(0, -3)$.

If $y = 0$:

$$\begin{aligned} \frac{-1}{2x+1} - 2 &= 0 \\ \frac{-1}{2x+1} &= 2 \\ -1 &= 2(2x+1) \\ -1 &= 4x+2 \\ 4x &= -3 \\ x &= -\frac{3}{4} \end{aligned}$$

The x -intercept is $(-\frac{3}{4}, 0)$

These facts alone should be sufficient to sketch the graph. It is a good idea to use both translations and these critical features when curve sketching.

Exercise B.4.4

1. State the asymptotes of these reciprocal functions and sketch their graphs:

i $y = \frac{1}{x-2}, x \neq 2$

ii $y = \frac{1}{x+4} - 3, x \neq -4$

iii $y = \frac{-3}{x-1} + 1, x \neq 1$

iv $y = \frac{2}{2x-3} + 2, x \neq 1.5$

v $y = \frac{-2}{2x-1} - 1, x \neq \frac{1}{2}$

vi $y = \frac{1}{1-2x} + 1, x \neq \frac{1}{2}$

- A hyperbola has asymptotes $x = 2$ and $y = -1$ and a line of symmetry parallel to $y = x$. Find its equation.
- A hyperbola has asymptotes $x = 1$ and $y = 1$ and passes through the points $(2, 3)$ & $(3, 2)$. Find its equation.
- State the equations of the lines of symmetry of the hyperbola:

$$y = \frac{2}{x-3} - 2, x \neq 3$$

- Find the lines of symmetry of $y = \frac{A}{x}, x \neq 0, A \in \mathbb{R}$.
- In an experiment, the volume of a gas (V litres) is tabulated against its pressure (P kiloPascals):

Pressure	38	45	53	62	78
Volume	12.4	10.4	8.9	7.6	6.0

Find a formula relating Pressure and Volume.

SL 2.8 The Rational Function

Rational functions take the form: $f(x) = \frac{ax+b}{cx+d}$

Graphs of this nature possess two types of asymptotes, one vertical and the other horizontal.

1. The vertical asymptote

A vertical asymptote occurs when the denominator is zero, that is, where $cx + d = 0$. Where this occurs, we place a vertical line (usually dashed), indicating that the curve cannot cross this line under any circumstances. This must be the case, because the function is undefined for that value of x .

For example, the function

$$x \mapsto \frac{3x+1}{2x+4}$$

is undefined for that value of x where $2x + 4 = 0$. That is, the function is undefined for $x = -2$. This means that we would need to draw a vertical asymptote at $x = -2$. In this case, we say that the asymptote is defined by the equation $x = -2$.

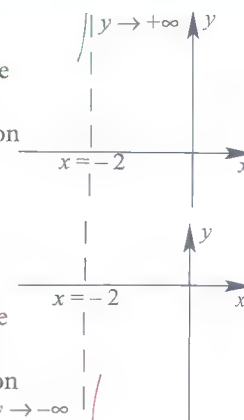
Using limiting arguments provides a more formal approach to 'deriving' the equation of the vertical asymptote. The argument is based along the following lines:

$$\text{as } x \rightarrow -2^-, \frac{3x+1}{2x+4} \rightarrow +\infty$$

That is, as x tends to -2 from the left or 'below', (hence the minus sign next to the two) the function tends to positive infinity.

$$\text{as } x \rightarrow -2^+, \frac{3x+1}{2x+4} \rightarrow -\infty$$

That is, as x tends to -2 from the right or 'above', (hence the plus sign next to the two) the function tends to negative infinity.



Therefore we write:

$$\left. \begin{array}{l} \text{As } x \rightarrow -2^+ \quad f(x) \rightarrow -\infty \\ \text{As } x \rightarrow -2^- \quad f(x) \rightarrow +\infty \end{array} \right\} \therefore x = -2 \text{ is a vertical asymptote}$$

$$\text{of } f(x) = \frac{3x+1}{2x+4}, x \neq -2.$$

2. The horizontal asymptote

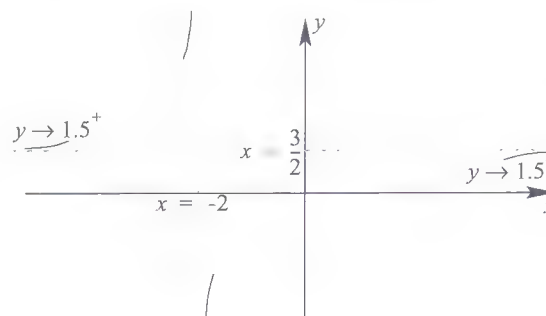
To determine the equation of the horizontal asymptote, we use a limiting argument, however, this time we observe the behaviour of the function as $x \rightarrow \pm\infty$.

It will be easier to determine the behaviour of the function (as $x \rightarrow \pm\infty$) if we first 'simplify' the rational function (using long division):

$$f(x) = \frac{3x+1}{2x+4} = \frac{3}{2} - \frac{5}{2x+4}.$$

Next we determine the behaviour for extreme values of x .

$$\left. \begin{array}{l} \text{As } x \rightarrow +\infty \quad f(x) \rightarrow \left(\frac{3}{2}\right)^- \\ \text{As } x \rightarrow -\infty \quad f(x) \rightarrow \left(\frac{3}{2}\right)^+ \end{array} \right\} \text{Therefore, } y = \frac{3}{2} \text{ is the horizontal asymptote}$$



We can now add a few more features of the function:

3. Axial intercepts

x -intercept

To determine the x -intercept(s) we need to solve for $f(x) = 0$.

In this case we have:

$$f(x) = \frac{3x+1}{2x+4} = 0 \Leftrightarrow 3x+1 = 0 \Leftrightarrow x = -\frac{1}{3}.$$

That is, the curve passes through the point $(-\frac{1}{3}, 0)$.

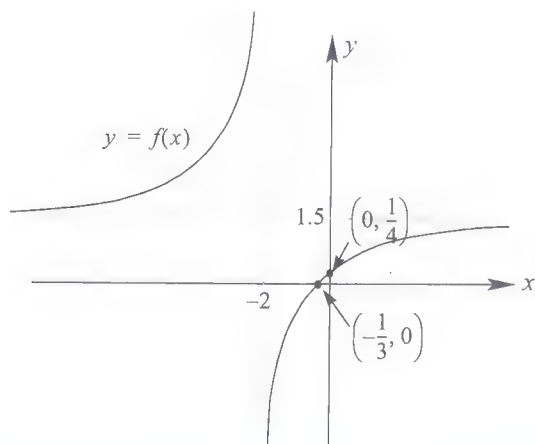
y -intercept

To determine the y -intercept we find the value of $f(0)$ (if it exists, for it could be that the line $x = 0$ is a vertical asymptote).

$$\text{In this case we have } f(0) = \frac{3 \times 0 + 1}{2 \times 0 + 4} = \frac{1}{4}.$$

Therefore the curve passes through the point $(0, \frac{1}{4})$.

Having determined the behaviour of the curve near its asymptotes (i.e. if the curve approaches the asymptotes from above or below) and the axial-intercept, all that remains is to find the stationary points (if any).



Example B.4.8

Sketch the graph of: $y = \frac{3x+1}{1-x}$

asymptote (e.g. at $x = -1.1$) the numerator is negative and the denominator small and negative. The y value is large and positive. Just to the right of the asymptote (e.g. at $x = -0.9$) the numerator is negative and the denominator small and positive. The y value is large and negative.

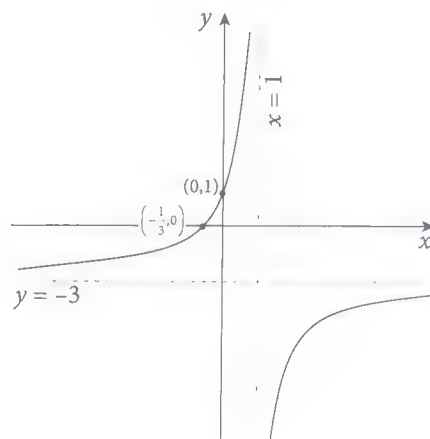
The horizontal asymptote occurs when x is large. When this happens, y tends to -3 . For a big negative x (e.g. $x = -100$), $y = -299/101$ i.e. a bit smaller than -3 (above the asymptote). For a big positive x (e.g. $x = 100$), $y = 301/99$ i.e. a bit bigger than -3 (below the asymptote).

Intercepts:

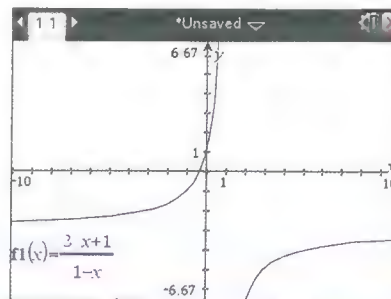
$$x \text{ intercept at } 3x+1=0 \Rightarrow x = -\frac{1}{3}$$

$$y \text{ intercept at } y = \frac{3 \times 0 + 1}{1 - 0} = 1$$

The sketch should show all the important features:



Modern graphic calculators will produce quite good plots of rational functions. They do not, however, show the key features which a good sketch must include.



The vertical asymptote is at $x = 1$. Just to the left of the

Exercise B.4.5

1. Use a limiting argument to determine the equations of the vertical and horizontal asymptotes for the following.

a $f(x) = \frac{2x+1}{x+1}$

b $f(x) = \frac{3x+2}{3x+1}$

c $f(x) = \frac{2x-1}{4x+1}$

d $f(x) = \frac{4-x}{x+3}$

e $f(x) = 3 - \frac{1}{x}$

f $f(x) = 5 - \frac{1}{2-x}$

2. Make use of a graphics calculator to verify your results from Question 1 by sketching the graph of the given functions.

3. Sketch the following curves, clearly labelling all intercepts, stating the equations of all asymptotes, and, in each case, showing that there are no stationary points.

a $x \mapsto \frac{3}{2x+1}$

b $x \mapsto \frac{x+1}{x+2}$

c $x \mapsto \frac{5-x}{2x-1}$

d $x \mapsto 3 + \frac{1}{x}$

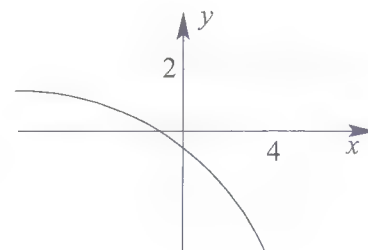
e $x \mapsto \frac{1}{x-3} - 2$

f $x \mapsto 1 - \frac{2}{2x-3}$

4. The figure shows part of the graph of the function whose equation is:

$$x \mapsto \frac{ax+2}{x-c}$$

Find the values of a and c .



5. Given that $f: x \mapsto x+2$ and that $g: x \mapsto \frac{1}{x-1}$, sketch the graphs of:

a $f \circ g$ b $g \circ f$

Extra questions



Summary

The reciprocal function has two distinctive features.

1. It is self-inverse.
2. It is discontinuous.

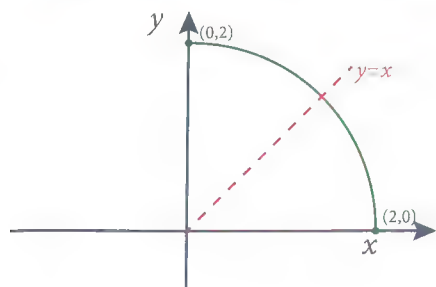
Self-inverse Functions

Are there any self-inverse functions other than $y = \frac{1}{x}, x \neq 0$?

These are symmetric about the line $y = x$, so $y = x$ is an example.

How about $y = -x + a$ for any a ?

More elaborate are functions such as: $y = \sqrt{4-x^2}, 0 \leq x \leq 2$ are also self inverse.



Discontinuous Functions

The cost of a taxi ride is usually calculated by a meter. This contains a function that records the time of the ride and the distance covered. If you watch the meter you will see that the costs recorded change suddenly - the readings are discontinuous.

The oddest of all discontinuous functions is probably this:

$$y = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}, 0 \leq x \leq 1$$

This is discontinuous everywhere in its domain.

The reason is that between any pair of rational numbers there is an irrational number and between any pair of irrational numbers there is a rational number

This function is a very strange beast!

SL 2.9 The Exponential Function

The **exponential function** takes the form:

$$f(x) = a^x, x \in \mathbb{R}, a > 0, a \neq 1$$

where the independent variable is the exponent.

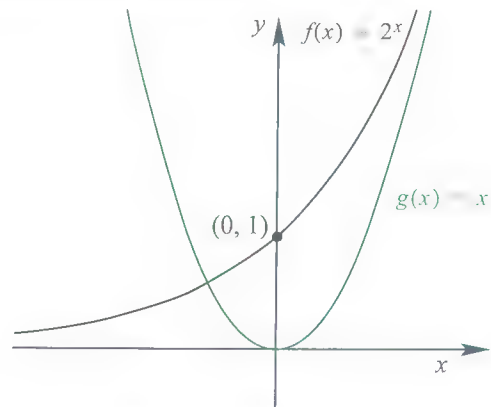
Graphs with $a > 1$

An example of an exponential function is $f(x) = 2^x, x \in \mathbb{R}$. So, how does the graph of $f(x) = 2^x$ compare with that of $f(x) = x^2$?

We know that the graph of $f(x) = x^2$ represents a parabola with its vertex at the origin, and is symmetrical about the y-axis. To determine the properties of the exponential function we set up a table of values and use these values to sketch a graph of $f(x) = 2^x$.

x	...	-2	-1	0	1	2	...
$g(x) = x^2$	...	4	1	0	1	4	...
$f(x) = 2^x$	...	$1/4$	$1/2$	1	2	4	...

We can now plot both graphs on the same set of axes and compare their properties:



Notice how different the graphs of the two functions are, even though their rules appear similar. The difference being that for the quadratic function, the variable x is the base, whereas for the exponential, the variable x is the power.

Properties of $f(x) = 2^x$

The function increases for all values of x (i.e. as x increases so too do the values of y).

The function is always positive (i.e. it lies above the x -axis).

As $x \rightarrow \infty$ then $y \rightarrow \infty$

$x \rightarrow -\infty$ then $y \rightarrow 0$. i.e. the x -axis is an asymptote.

When $x > 0$ then $y > 1$,

$x = 0$ then $y = 1$

$x < 0$ then $0 < y < 1$.

We can now investigate the exponential function for different bases. Consider the exponential functions $f(x) = 4^x$ and $g(x) = 3^x$. From the graphs we can see that $f(x) = 4^x$ increases much faster than $g(x) = 3^x$ for $x > 0$.

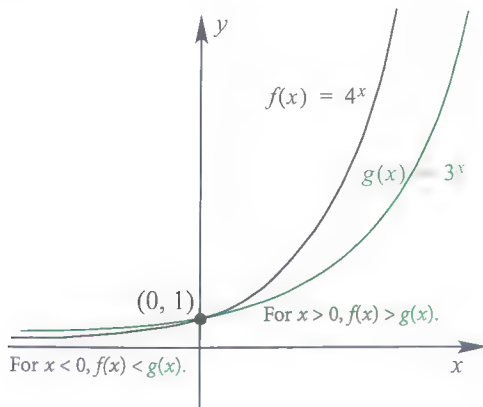
For example, at $x = 1$, $f(1) = 4$, $g(1) = 3$ and then, at $x = 2$, $f(2) = 16$, $g(2) = 9$. However, for $x < 0$ we have, $f(x) = 4^x$ decreases faster than $g(x) = 3^x$.

Notice then that at $x = 0$, both graphs pass through the point $(0, 1)$.

From the graphs we can see that for values of x less than zero, the graph of $f(x) = 4^x$ lies below that of $g(x) = 3^x$. Whereas for

values of x greater than zero, then the graph of $f(x) = 4^x$ lies above that of $g(x) = 3^x$.

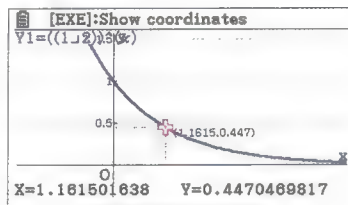
Exponential functions that display these properties are referred to as exponential growth functions.



What happens when $0 < a < 1$?

Consider the case where $a = \frac{1}{2}$.

Rather than using a table of values we provide a sketch of the curve (opposite). The graph shows that the function is decreasing – such exponential functions are referred to as exponential decay.



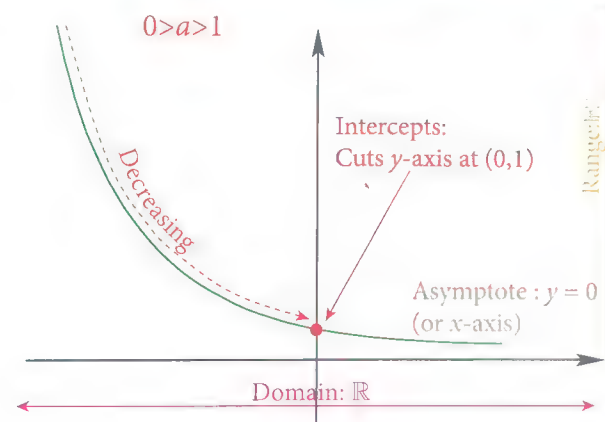
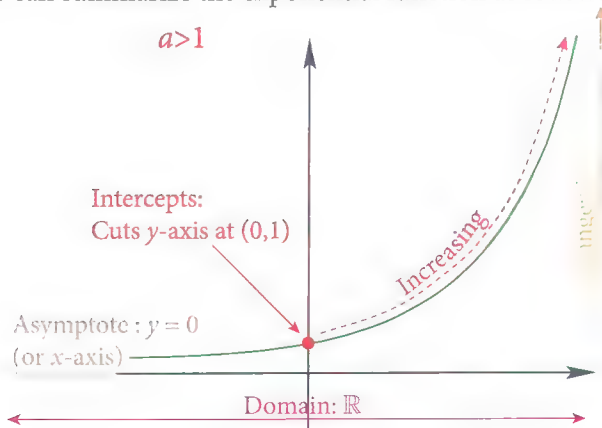
In fact, from the screen-shot we can see that the graph of $y = (\frac{1}{2})^x$ is a reflection of $y = 2^x$ about the y -axis.

We note that the function $y = (\frac{1}{2})^x$ can also be written as $y = (2^{-1})^x = 2^{-x}$.

There are two ways to represent an exponential decay function, either as $f(x) = a^x$, $0 < a < 1$ or $f(x) = a^{-x}$, $a > 1$.

For example, the functions $f(x) = (\frac{1}{4})^x$ and $g(x) = 4^{-x}$ are identical.

We can summarize the exponential function as follows:



There also exists an important exponential function known as the natural exponential function.

This function is such that the base has the special number 'e'. The number 'e', has a value that is given by the expression:

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \approx 2.71828...$$

However, at this stage it suffices to realise that the number 'e' is greater than one. This means that a function of the form $f(x) = e^x$ will have the same properties as that of $f(x) = a^x$ for $a > 1$. That is, it will depict an exponential growth. Whereas the function $f(x) = e^{-x}$ will depict an exponential decay.

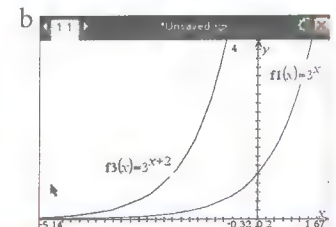
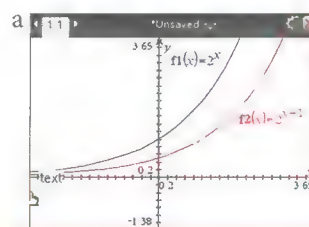
Example B.4.9

On the same set of axes sketch the following.

a $f(x) = 2^x, g(x) = 2^{x-1}$

b $f(x) = 3^x, g(x) = 3^{x+2}$

Using a calculator:



Notice that the principles of transformations discussed in the previous section elsewhere apply to exponential graphs. In Example B.4.8 a we have a 'right one unit' translation and in b, a 'left two units' translation.

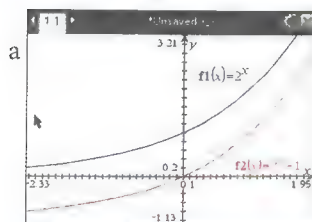
Example B.4.10

On the same set of axes sketch the following.

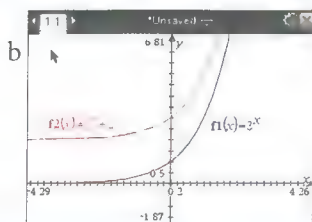
a $f(x) = 2^x, g(x) = 2^x - 1$

b $f(x) = 3^x, g(x) = 3^x + 2$

Using a calculator:

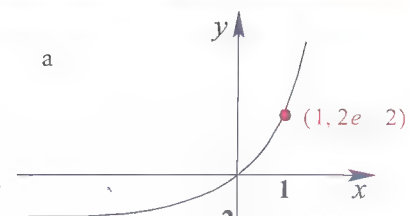


'down 1'

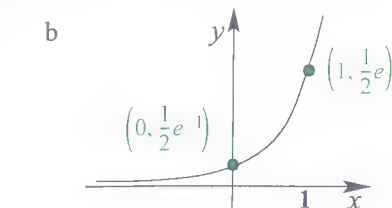


'up 2'

Notice that in both cases the general shape of the exponential growth remains unaltered. Only the main features of the graph are of interest when sketching is involved.



When $x = 0, f(0) = 2 \times e^0 - 2 = 0$.
When $x = 1, f(1) = 2 \times e^1 - 2$.



When $x = 0, f(0) = \frac{1}{2}e^{0-1} = \frac{1}{2}e^{-1}$.

When $x = 1, f(1) = \frac{1}{2}e^{2-1} = \frac{1}{2}e^1$.

Exercise B.4.6

- On separate sets of axes, sketch the graphs of the following functions and determine the range of each function.

a $f(x) = 4^x$

b $f(x) = 3^x$

c $f(x) = 5^x$

d $f(x) = (2.5)^x$

e $f(x) = (3.2)^x$

f $f(x) = (1.8)^x$

g $f(x) = \left(\frac{1}{2}\right)^x$

h $f(x) = \left(\frac{1}{3}\right)^x$

i $f(x) = \left(\frac{1}{5}\right)^x$

j $f(x) = \left(\frac{3}{4}\right)^x$

k $f(x) = \left(\frac{5}{8}\right)^x$

l $f(x) = (0.7)^x$

- Sketch the following on the same set of axes, clearly labelling the y -intercept.

$f(x) = 3^x + c$ where i $c = 1$ ii $c = -2$

$f(x) = 2^{-x} + c$ where i $c = 0.5$ ii $c = -0.5$

- Sketch the following on the same set of axes, clearly labelling the y -intercept.

$f(x) = b \times 3^x$ where i $b = 2$ ii $b = -2$

$f(x) = b \times \left(\frac{1}{2}\right)^x$ where i $b = 3$ ii $b = -2$

Example B.4.11

On the same set of axes sketch the following.

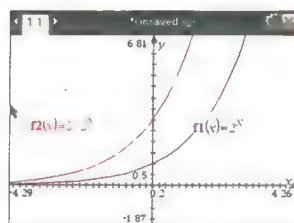
a $f(x) = 2^x, g(x) = 3 \times 2^x$

b $f(x) = 3^x, g(x) = -\left(\frac{1}{2}\right) \times 3^x$

Using a calculator:

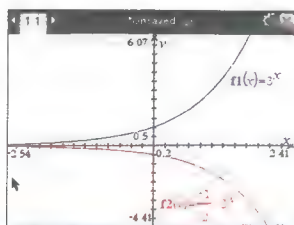
a

Vertical stretch factor 3.



b

Reflection in x -axis and dilation of $\frac{1}{2}$ in the y direction.



Example B.4.12

Sketch the following.

a $f(x) = 2 \times e^x - 2$

b $f(x) = \frac{1}{2}e^{2x-1}$

4. On the same set of axes, sketch the following graphs.

a $f(x) = 3^x$ and $f(x) = 3^{-x}$

b $f(x) = 5^x$ and $f(x) = 5^{-x}$

c $f(x) = 10^x$ and $f(x) = 10^{-x}$

d $f(x) = \left(\frac{1}{3}\right)^x$ and $f(x) = \left(\frac{1}{3}\right)^{-x}$

5. Find the ranges of the following functions.

a $f: [0, 4] \rightarrow \mathbb{R}, y = 2^x$

b $f: [1, 3] \rightarrow \mathbb{R}, y = 3^x$

c $f: [-1, 2] \rightarrow \mathbb{R}, y = 4^x$

d $f: [-1, 2] \rightarrow \mathbb{R}, y = 2^x$

e $f: [2, 3] \rightarrow \mathbb{R}, y = 2^{-x}$

f $f: [-1, 1] \rightarrow \mathbb{R}, y = 10^{-x}$

6. Sketch the graphs of the following functions, stating their range.

a $f: \mathbb{R} \rightarrow \mathbb{R}$ where $f(x) = 2e^x + 1$

b $f: \mathbb{R} \rightarrow \mathbb{R}$ where $f(x) = 3 - e^{x-1}$

c $f: \mathbb{R} \rightarrow \mathbb{R}$ where $f(x) = e - e^{-x}$

d $f: \mathbb{R} \rightarrow \mathbb{R}$ where $f(x) = 2 + \frac{1}{2}e^{-x}$

Extra questions



The Logarithmic Function

The logarithmic function, with base 'a' is represented by the expression $g(x) = \log_a x, x > 0$.

The logarithmic function is the inverse of the exponential function:

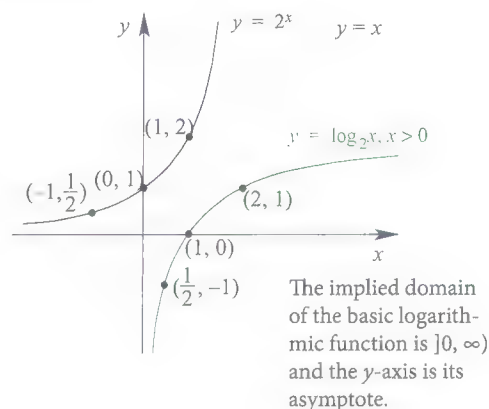
$$\text{If } f(x) = 2^x \text{ then } f^{-1}(x) = \log_2 x$$

To determine the shape of its graph we start by constructing a table of values for the function $f^{-1}(x) = \log_2 x$ and comparing it with the table of values for $f(x) = 2^x$:

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	0	1	2
$y = \log_2 x$	-3	-2	-1	-	0	1

x	-3	-2	-1	0	1	2
$y = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4

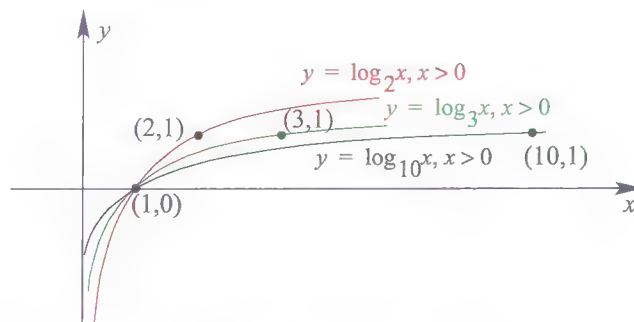
From the table of values we observe that the x - and y -values have interchanged! Plotting these results on the same set of axes, we observe that the graph of the logarithmic function is a reflection of the exponential function about the line $y = x$. So, whereas for the exponential function, the asymptote is the x -axis (i.e. $y = 0$), for the logarithmic function, the asymptote is the y -axis (i.e. $x = 0$).



So, how do the graphs of $y = \log_e x$, $y = \log_{10} x$, $y = \log_3 x$ compare to $y = \log_2 x$? The best way to see this is to sketch the graphs on the same set of axes as the diagram shows:

The implied domain of the basic logarithmic function with any positive base is $]0, \infty[$ and has the y -axis as its asymptote.

Observe that each of the logarithmic functions is a reflection about the line $y = x$ of its corresponding exponential function (of the same base).



Notice that for all $a > 0$ $\log_a 1 = 0$ and $\log_a a = 1$.

As is the case for the exponential functions, the base 'e' also plays an important role when dealing with logarithmic functions. When using the number 'e' as the base for the logarithmic function, we refer to it as the **natural logarithmic function** and can write it in one of two ways:

$$f(x) = \log_e x, x > 0 \text{ or } f(x) = \ln x, x > 0$$

Example B.4.13

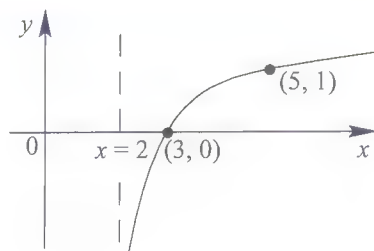
Sketch the following, specifying the implied domain in each case.

a $f(x) = \log_3(x-2)$ b $g(x) = \log_2(2x+3)$

a

A '2 right' translation.

Domain of $f = (2, \infty)$.



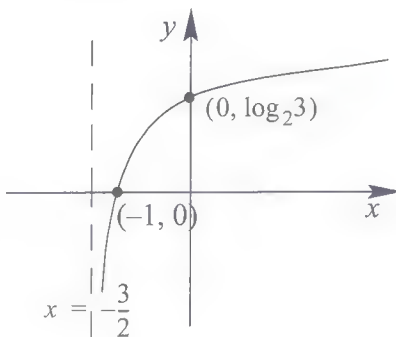
b

Looking at the domain:

$$2x+3 > 0 \Leftrightarrow x > -\frac{3}{2}.$$

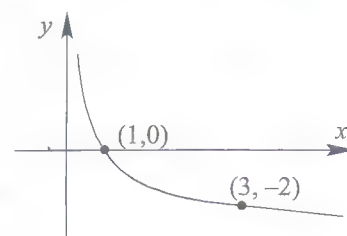
Therefore, the implied domain of $g = (-\frac{3}{2}, \infty)$.

The vertical asymptote is: $x = -\frac{3}{2}$.



a The implied domain in this case is $x > 0$. So, the vertical asymptote has the equation $x = 0$.

We note that the negative sign in front of the $\log_3 x$ will have the effect of reversing the sign of the $\log_3 x$ values.



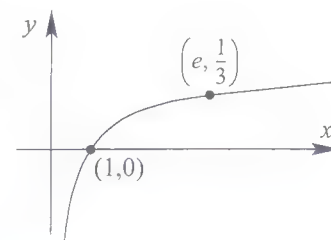
That is, the graph of $f(x) = -\log_3 x$ is a reflection about the x -axis of the graph of $y = \log_3 x$.

The factor of 2 will have the effect of 'stretching' the graph of $y = \log_3 x$ by a factor of 2 along the y -axis.

Also, we have that $f(1) = -2\log_3 1 = 0$ and $f(3) = -2\log_3 3 = -2$.

b This time the implied domain is $]0, \infty)$.

Therefore, the equation of the asymptote is $x = 0$. The one third factor in front of $\log_e x$ will have the effect of 'shrinking' the graph of $y = \log_e x$ by a factor of 3.



Then, $f(1) = \frac{1}{3}\log_e 1 = 0$ and $f(e) = \frac{1}{3}\log_e e = \frac{1}{3}$.

Again, we have the following observations:

The graph of $y = k \times \log_a x, k > 0$ is identical to $y = \log_a x$ but

- i stretched along the y -axis if $k > 1$.
- ii shrunk along the y -axis if $0 < k < 1$.

The graph of $y = k \times \log_a x, k < 0$ is identical to $y = \log_a x$ but

- i reflected about the x -axis and stretched along the y -axis if $k < -1$.
- ii reflected about the x -axis and shrunk along the y -axis if $-1 < k < 0$.

Video



Example B.4.14

Sketch the following, specifying the implied domain in each case.

a $f(x) = -2\log_3 x$ b $f(x) = \frac{1}{3}\log_e x$

Example B.4.15

Sketch the following, specifying the implied domain in each case.

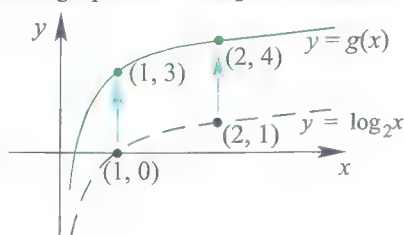
a $g(x) = \log_2 x + 3$

b $g(x) = \log_2 x - 2$

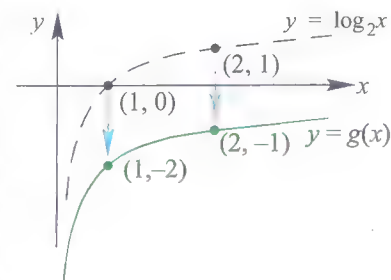
c $g(x) = \log_2(3 - x)$

a The effect of adding 3 to the graph of $y = \log_2 x$ will result in $g(x) = \log_2 x + 3$ being moved up 3 units.

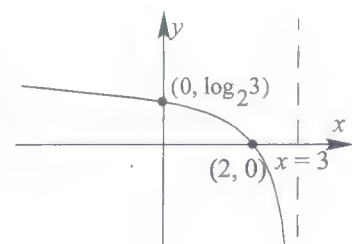
Its implied domain is $(0, \infty)$ and its asymptote has equation $x = 0$.



b The effect of subtracting 2 from the graph of $y = \log_2 x$ will result in $g(x) = \log_2 x - 2$ being moved down 2 units. Its implied domain is $(0, \infty)$ and its asymptote has equation $x = 0$.



c As we cannot have the logarithm of a negative number we must have that $3 - x > 0 \Leftrightarrow x < 3$.



This means that the vertical asymptote is given by $x = 3$ and the graph must be drawn to the left of the asymptote.

Exercise B.4.7

1. Sketch the graphs of the following functions, clearly stating domains and labelling asymptotes.

a $f(x) = \log_4(x - 2)$

b $f(x) = \log_2(x + 3)$

c $h(x) = \log_{10} x + 2$

d $g(x) = -3 + \log_3 x$

e $f(x) = \log_5(2x - 1)$

f $h(x) = \log_2(2 - x)$

g $g(x) = 2\log_{10} x$

h $f(x) = -\log_{10} x + 1$

2. Sketch the graphs of the following functions, clearly stating domains and labelling asymptotes.

a $f(x) = 2\log_2 x + 3$

b $f(x) = 10 - 2\log_{10} x$

c $h(x) = 2\log_2 2(x - 1)$

d $g(x) = -\frac{1}{2}\log_{10}(1 - x)$

e $f(x) = \log_2(3x + 2) - 1$

f $h(x) = 3\log_2\left(\frac{1}{2}x - 1\right) + 1$

3. Sketch the graphs of the following functions, clearly stating domains and labelling asymptotes.

a $f(x) = 2\ln x$

b $g(x) = -5\ln x$

c $f(x) = \ln(x - e)$

d $f(x) = \ln(1 - ex)$

e $f(x) = 5 - \ln x$

f $h(x) = \ln x - e$

4. Sketch the graphs of the following functions, clearly stating domains and labelling asymptotes.

a $f(x) = \log_2 \sqrt{x}$

b $f(x) = \log_{10} x^2$

c $h(x) = \ln|x|$

d $g(x) = \ln\left(\frac{1}{x}\right)$

e $h(x) = \ln(1 - x^2)$

f $f(x) = \log_2(x^2 - 4)$

5. Sketch the graphs of the following functions, clearly stating domains and labelling asymptotes.

a $f(x) = |\log_{10} x|$

b $g(x) = |\log_2(x - 1)|$

c $h(x) = |\ln x - 1|$

d $h(x) = 2 - |\ln x|$

e $f(x) = \log_2|x + 2|$

f $f(x) = \log_5\left|\frac{1}{x}\right|$

Extra questions

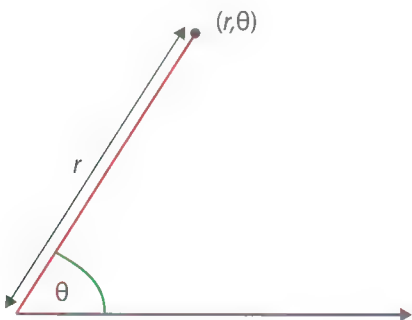




The Logarithmic Spiral

As we observe elsewhere, the logarithmic spiral is found in nature. The shape appears to arise naturally from the way organisms grow. If you wish to pick this up as your investigation topic, you will find plenty of examples from both flora and fauna.

The spiral is defined by polar coordinates. These are not on this course, but do represent a reasonable extension of your studies. Whilst cartesian coordinates specify points by assigning number pairs to each using a rectangular grid, polar coordinates work rather like radar. The point is identified by its distance from the origin and its angular deflection from a zero direction.



Rather in the way curves in the cartesian plane are defined by equations relating x and y , radial curves are defined by equations relating r and θ .

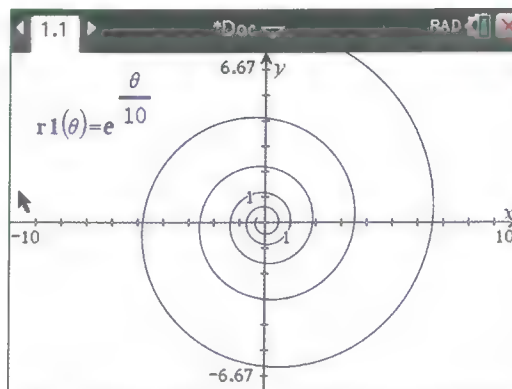
The logarithmic spiral is defined by $r = ae^{b\theta}$ where a and b are parameters that determine the exact shape of the curve.

This can be rearranged using techniques mentioned earlier.

$$\frac{r}{a} = e^{b\theta} \Rightarrow \ln\left(\frac{r}{a}\right) = b\theta \Rightarrow \theta = \frac{1}{b} \ln\left(\frac{r}{a}\right)$$

This last expression gives the curve its name.

Here is an example as implemented on a graphic calculator.

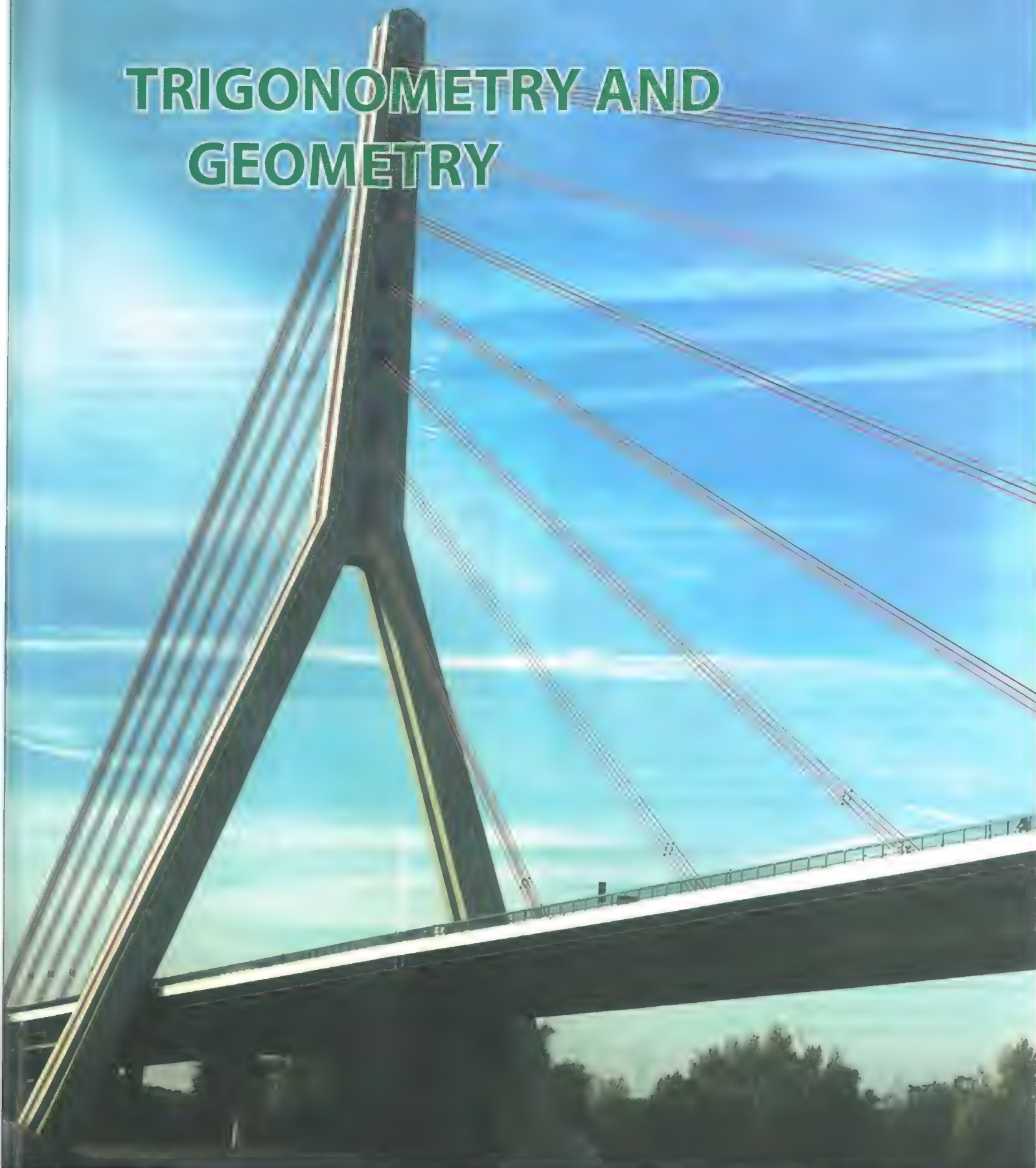


Answers



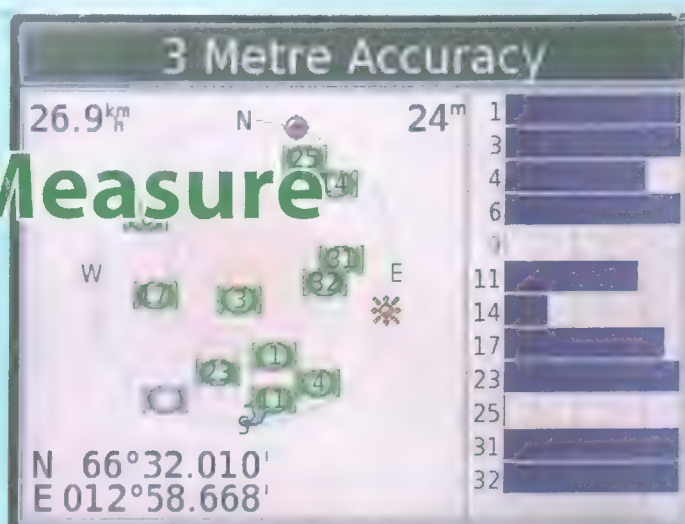
SECTION THREE

TRIGONOMETRY AND GEOMETRY





C.4 Radian Measure



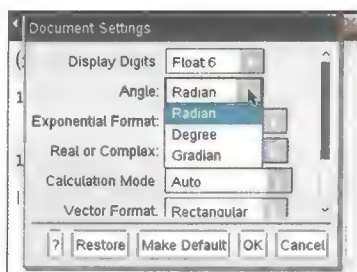
GARMIN nüvi

Our cover picture shows a modern GPS unit as it crosses the Arctic Circle - and the geographic marker that confirms it. It all depends on angles!

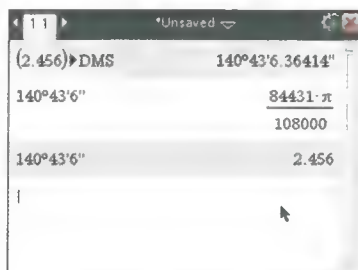
Radian Measure of an Angle

In Middle School Mathematics angles are measured in degrees. However, while this has been very useful, such measurements are not suitable for many topics in mathematics. Instead, we introduce a new measure, called the **radian** measure.

The degree measure of an angle is based on dividing the complete circle into 360 equal parts known as degrees. Each degree is divided into sixty smaller parts known as minutes, and each minute is divided into sixty seconds. If using a calculator, you need to know how to set the calculator to radians or degrees. This is TI.



Decimal parts of a degree can be converted into degrees, minutes and seconds using the book key (to the right of the 9, pressing D to scroll quickly to DMS and then selecting the function. The calculator should be in degree mode.



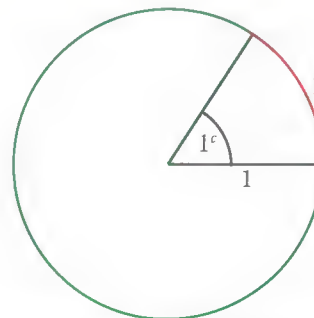
This can be useful as calculators generally produce answers in the decimal format. It should also be noted that the degree,

minute, second angle system is the same as the hours minutes seconds system that we use to measure time. The previous screen could be interpreted as 2.456° and is equal to 2 degrees 27 minutes and 21.6 seconds or as 2.456 hours which is the same as 2 hours 27 minutes and 21.6 seconds. Note also that there is a difference between 'ENTER' and 'CTRL/ENTER'.

The degree system is arbitrary in the sense that the decision was made (in the past and due to astronomical measurements) to divide the complete circle into 360 parts. The radian system is an example of a natural measurement system.

One radian is defined as the ratio between the arc and the radius of the circle.

Two radians is the angle that gives an arc length of twice the radius etc. giving a natural linear conversion between the measure of a radian, the arc length and the radius of a circle.



A complete circle has an arc length of $2\pi r$.

It follows that a complete circle corresponds to $\frac{2\pi r}{r} = 2\pi$ radians.

This leads to the conversion factor between these two systems:

$$360^\circ = 2\pi \text{ radians or } 180^\circ = \pi \text{ radians (often written as } \pi^c).$$

So, exactly how large is a radian?

Using the conversion above, if $360^\circ = 2\pi^c$, then $1^c = \frac{360}{2\pi} \approx 57.2957^\circ$

That is, the angle which subtends an **arc of length 1 unit** in a **circle of radius 1 unit**, is **1 radian**.

More generally we have:

To convert from degrees to radians, multiply angle by $\frac{\pi}{180}$
To convert from radians to degrees, multiply angle by $\frac{180}{\pi}$

All conversions between the two systems follow this ratio. It is not generally necessary to convert between the systems as problems are usually worked either entirely in the degree system (as in the previous sections) or in radians (as in the functions and calculus chapters). In the case of arc length and sector areas, it is generally better to work in the radian system.

Example C.4.1

Convert:

- 70° into radians
- 2.34^c into degrees
- $\frac{\pi^c}{6}$ into degrees

Using the above conversion factors we have:

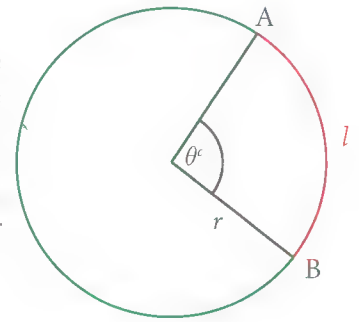
$$a \quad 70^\circ = 70 \times \frac{\pi^c}{180} = \frac{7\pi^c}{18} \text{ or } 1.2217^c.$$

$$b \quad 2.34^c = 2.34 \times \frac{180^\circ}{\pi} = 134.0721^\circ = 134^\circ 4' 20''$$

$$c \quad \frac{\pi^c}{6} = \frac{\pi}{6} \times \frac{180^\circ}{\pi} = 30^\circ$$

Arc length

As the arc length AB of a circle is directly proportional to the angle which AB subtends at its centre, then, the arc length AB is a fraction of the circumference of the circle of radius r .



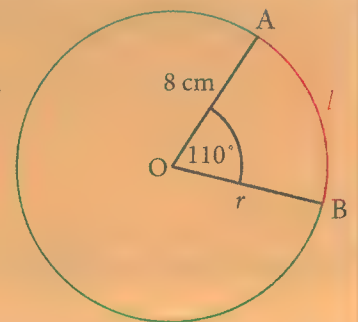
So, if the angle is θ^c , then the arc length is $\frac{\theta}{2\pi}$ of the circumference.

Then, the (minor) arc length, AB, denoted by l , is given by $l = \frac{\theta}{2\pi} \times 2\pi r = r\theta$.

The longer arc AB, called the **major arc**, has a length of $2\pi r - l$.

Example C.4.2

Using the circle shown, find the arc length AB.



First we need to convert 110° into radian measure.

$$110^\circ = 110 \times \frac{\pi^c}{180} = \frac{11\pi^c}{18}$$

Then, the arc length, l , is given by,

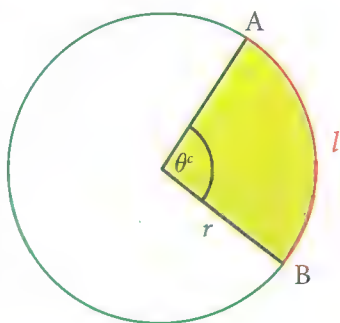
$$l = r\theta = 8 \times \frac{11\pi}{18} = \frac{44\pi}{9} = 15.3588...$$

Therefore, the arc length is 15.36 cm.

Area of a Sector

The formula for the area of a sector is derived as follows:

If a sector is cut from a circle of radius r using an angle at the centre of θ radians, the area of the complete circle is πr^2 .



The fraction of the circle that forms the sector is $\frac{\theta}{2\pi}$, so the area of the sector is:

$$\pi r^2 \times \frac{\theta}{2\pi} = \frac{1}{2} r^2 \theta.$$

$$\text{The angle of the shaded segment} = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$$

The shaded area can be found by subtracting the area of the sector in the smaller circle from that in the larger circle.

Shaded area =

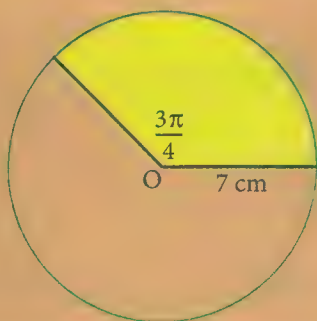
$$\frac{1}{2} \times 9^2 \times \frac{11\pi}{6} - \frac{1}{2} \times 4^2 \times \frac{11\pi}{6} = \frac{11\pi}{12} (9^2 - 4^2) = 59\frac{7}{12}\pi \text{ cm}^2$$

The perimeter is made up from two straight lines (each $9 - 4 = 5$ cm long) and two arcs.

$$\text{Perimeter} = 10 + 4 \times \frac{11\pi}{6} + 9 \times \frac{11\pi}{6} = 10 + \frac{143\pi}{6} \text{ cm.}$$

Example C.4.3

Find the area and perimeter of the sector shown.



$$\text{Area of sector} = \frac{1}{2} \times 7^2 \times \frac{3\pi}{4} = \frac{147\pi}{8} \text{ cm}^2.$$

The perimeter is made up from two radii (14 cm) and the arc $l = r\theta = 7 \times \frac{3\pi}{4} = \frac{21\pi}{4}$.

The perimeter is $14 + \frac{21\pi}{4}$ cm.

Exercise C.4.1

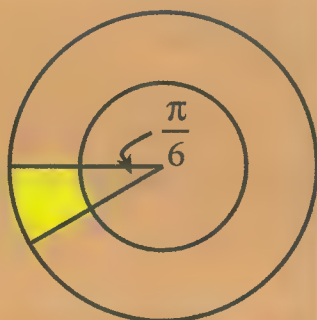
1. Find the areas and perimeters of the following sectors.

	Radius	Angle
a	2.6 cm	$\frac{\pi}{3}$
b	11.5 cm	$\frac{\pi}{4}$
c	44 cm	$\frac{\pi}{4}$
d	6.8 m	$\frac{2\pi}{3}$
e	0.64 cm	$\frac{3\pi}{4}$
f	7.6 cm	$\frac{5\pi}{6}$
g	324 m	$\frac{\pi}{10}$

2. A cake has a circumference of 30 cm and a uniform height of 7 cm. A slice is to be cut from the cake with two straight cuts meeting at the centre. If the slice is to contain 50 cm^3 of cake, find the angle between the two cuts, giving the answer in radians to 2 significant figures and in degrees correct to the nearest degree.

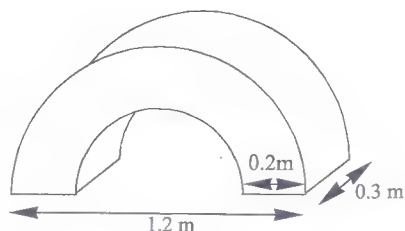
Example C.4.4

Find the area and perimeter of the shaded part of the diagram. The radius of the inner circle is 4 cm and the radius of the outer circle is 9 cm.

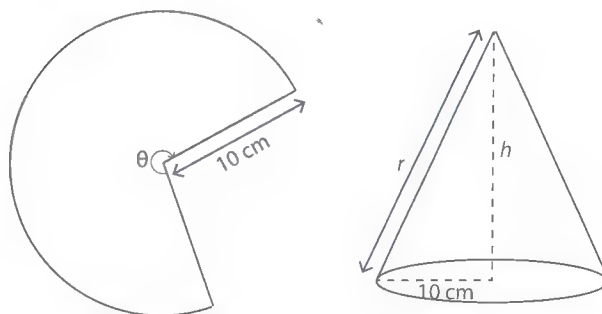


3. The diagram shows a part of a Norman arch. The dimensions are shown in metres.

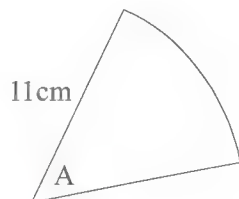
Find the volume of stone in the arch, giving your answer in cubic metres, correct to three significant figures.



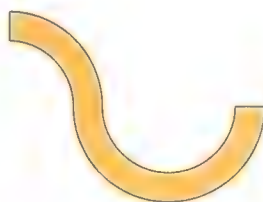
7. The diagrams show a circular sector of radius 10 cm and angle θ radians which is formed into a cone of slant height 10 cm. The vertical height h of the cone is equal to the radius r of its base. Find the angle θ radians.



4. In the diagram, find the value of the angle A in radians, correct to three significant figures, if the perimeter is equal to 40 cm.



5. The diagram shows a design for a shop sign. The arcs are each one quarter of a complete circle. The radius of the smaller circle is 7 cm and the radius of the larger circle is 9 cm.



Find the perimeter of the shape, correct to the nearest centimetre.

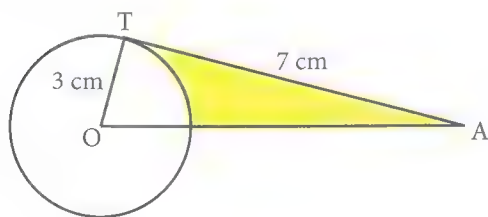
Extra questions



Answers



6. Find the shaded area in the diagram. The dimensions are given in centimetres. O is the centre of the circle and AT is a tangent.



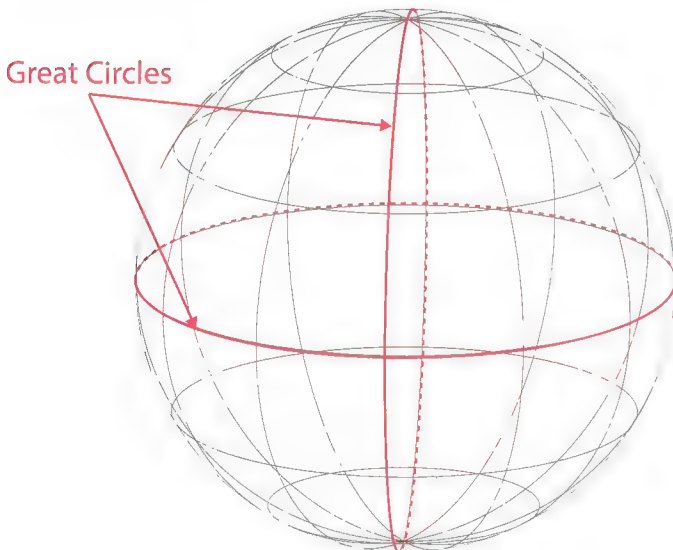
Give your answer correct to three significant figures.



Great Circles

The techniques discussed in this chapter are of great importance in marine and aviation navigation.

Unlike navigation on roads, where one has to follow pre-defined paths, crossing (for example) the Pacific Ocean in a ship leaves a choice of routes. The only real constraint is that one has to follow the curve of the Earth



The great circles we have shown are the equator and one line of longitude. Note that all the other lines of latitude are smaller.

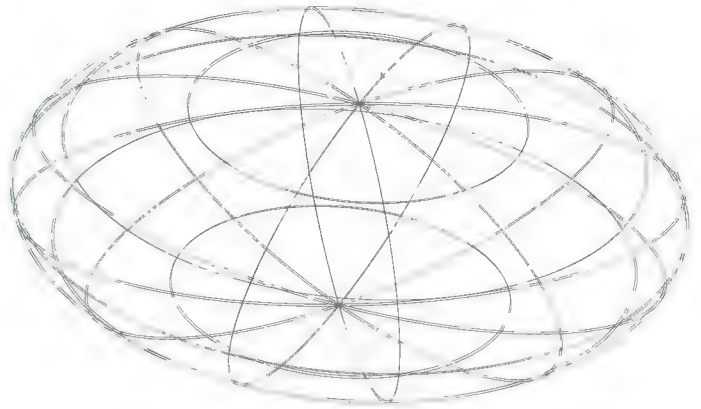
It is arcs of these great circles that define the shortest navigation routes.

Can you solve the problem of finding the shortest route between two points on the Earth? Like the fact that there is a single straight line joining two points on a plane, there is a single great circle joining two points on the Earth.

Can you, for example, find the shortest distance between the Eiffel Tower (48°51'29.6"N 2°17'40.2"E) and the Taj Mahal (27°10'30"N 78°02'31"E)?

You may ignore the fact that the Earth is not a perfect sphere.

The problem becomes even more challenging when applied to non-spherical 'planets' like the planet *Elliptica*.



Or the planet *Torus*:



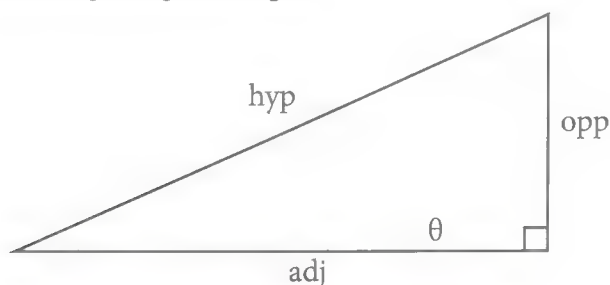
C.5 Identities

SL 3.5

SL 3.6

The Unit Circle

Middle School courses usually include how to find the sine, cosine and the tangent of acute angles contained within a right-angled triangle.



$$\sin \theta = \frac{\text{opp}}{\text{hyp}}, \cos \theta = \frac{\text{adj}}{\text{hyp}}, \tan \theta = \frac{\text{opp}}{\text{adj}}$$

We can extend this to enable us to find the sine and cosine ratio of obtuse angles. To see why this works, or indeed why it would work for an angle of any magnitude, we need to reconsider how angles are measured. To do this we start by making use of the unit circle and introduce some definitions.

From this point on we define the angle θ as a real number that is measured in either degrees or radians. So that, an expression such as $\sin(180^\circ - \theta)$ will imply that θ is measured in degrees as opposed to the expression $\sin(\pi^\circ - \theta)$ which would imply that θ is measured in radians. In both cases, it should be clear from the context of the question which one it is.

Exact Values

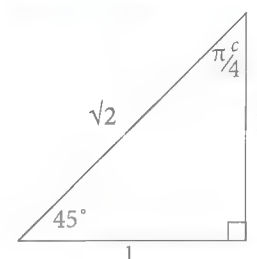
There are several 'exact values' of acute angles that are important. They result from two special triangles. It is probably easier to remember the triangles rather than a table of values.

Isosceles Right Triangle

$$\sin 45^\circ = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\cos 45^\circ = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\tan 45^\circ = \tan \frac{\pi}{4} = 1$$



Right Triangle in an Equilateral Triangle

$$\sin 30^\circ = \sin \frac{\pi}{6} = \frac{1}{2}$$

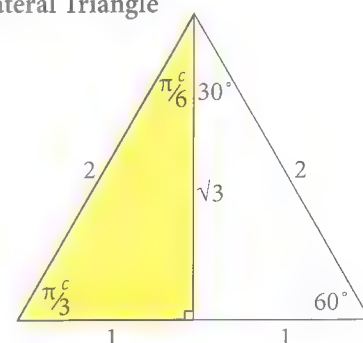
$$\cos 30^\circ = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

$$\sin 60^\circ = \sin \frac{\pi}{3} = \frac{1}{2} = \frac{\sqrt{3}}{2}$$

$$\cos 60^\circ = \cos \frac{\pi}{3} = \frac{1}{2}$$

$$\tan 60^\circ = \tan \frac{\pi}{3} = \sqrt{3}$$



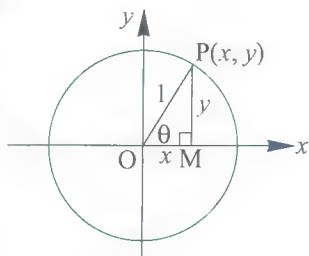
Finally, we have the extreme values of 0° and 90° which are not really representable in a right triangle.

$$\sin 0^\circ = 0, \cos 0^\circ = 1, \tan 0^\circ = 0$$

$$\sin 90^\circ = 1, \cos 90^\circ = 0, \tan 90^\circ \text{ is undefined}$$

Note that $\tan 90^\circ$ is undefined. We will shortly see why this is the case.

By convention, an angle θ is measured in terms of the rotation of a ray OP from the positive direction of the x -axis, so that a rotation in the **anticlockwise** direction is described as a **positive** angle, whereas a rotation in the **clockwise** direction is described as a **negative** angle.



Let the point $P(x, y)$ be a point on the circumference of the unit circle, $x^2 + y^2 = 1$, with centre at the origin and radius 1 unit.

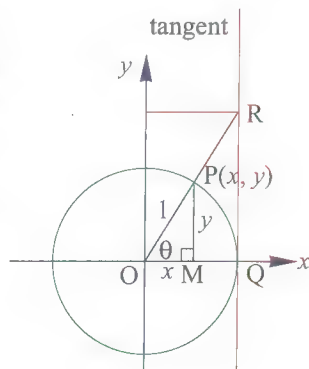
With OP making an angle of θ with the positive direction of the x -axis, we draw the perpendicular from P to meet the x -axis at M. This then provides the following definitions:

$$\begin{aligned}\sin \theta &= \frac{MP}{OP} = \frac{y\text{-coordinate of } P}{OP} = \frac{y}{1} = y \\ \cos \theta &= \frac{OM}{OP} = \frac{x\text{-coordinate of } P}{OP} = \frac{x}{1} = x \\ \tan \theta &= \frac{MP}{OM} = \frac{y\text{-coordinate of } P}{x\text{-coordinate of } P} = \frac{y}{x}\end{aligned}$$

Note that this means that the y -coordinate corresponds to the **sine** of the angle θ , that the x -coordinate corresponds to the **cosine** of the angle θ and that the tangent, . . . , well, for the tangent, let's revisit the unit circle, but this time we will make an addition to the diagram.

Using the existing unit circle, we draw a tangent at the point where the circle cuts the positive x -axis, Q.

Next, we extend the ray OP to meet the tangent at R.



Using similar triangles, we have that $\frac{PM}{OM} = \frac{RQ}{OQ} = \frac{RQ}{1}$.

That is, $\tan \theta = RQ$ - which means that the value of the tangent of the angle θ corresponds to the y -coordinates of point R cut off on the tangent at Q by the extended ray OP.

Also, it is worth noting that $\tan \theta = \frac{PM}{OM} = \frac{y}{x} = \frac{\sin \theta}{\cos \theta}$ (as long as $\cos \theta \neq 0$).

That is, $\tan \theta = \frac{\sin \theta}{\cos \theta}, \cos \theta \neq 0$.

From our list of exact values, we note that $\tan 90^\circ$ is undefined. This can be observed from the previous diagram. If $\theta = 90^\circ$, P lies on the y -axis, meaning that OP would be parallel to QR, and so, P would never cut the tangent, meaning that no y -value corresponding to R could ever be obtained.

Angle of any Magnitude

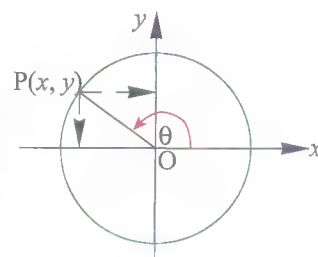
From the unit circle we have seen how the trigonometric ratios of an acute angle can be obtained, i.e. for the sine ratio we read off the y -axis, for the cosine ratio, we read off the x -axis and for the tangent ratio we read off the tangent. As the point P is located in the first quadrant, then $x \geq 0$, $y \geq 0$ and $\frac{y}{x} \geq 0$, $x \neq 0$.

This means that we obtain positive trigonometric ratios.

So, what if P lies in the second quadrant?

We start by drawing a diagram for such a situation:

From our diagram we see that if P lies in the second quadrant, the y -value is still positive, the x -value is negative and therefore the ratio, $\frac{y}{x}$ is negative.



This means that, $\sin \theta > 0$, $\cos \theta < 0$ and $\tan \theta < 0$.

In a similar way, we can conclude that if $180^\circ < \theta < 270^\circ$, i.e. the point P is in the third quadrant, then,

y -value is negative $\sin \theta < 0$

x -value is negative $\cos \theta < 0$

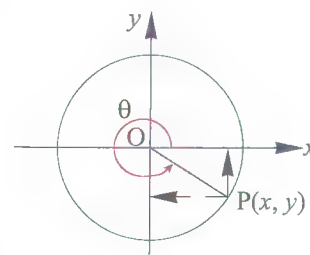
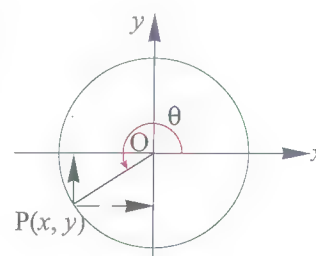
the quotient is positive $\tan \theta > 0$

For the **fourth quadrant** we have, $270^\circ < \theta < 360^\circ$, so that:

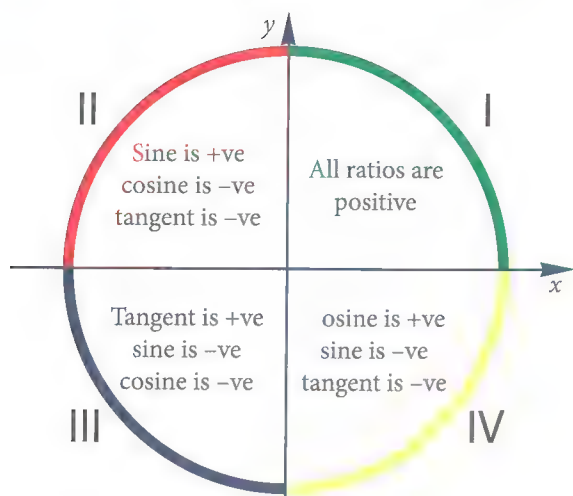
y -value is negative $\sin \theta < 0$

x -value is positive $\cos \theta > 0$

the quotient is negative $\tan \theta < 0$



We now know that, depending on which quadrant an angle lies in, the sign of the trigonometric ratio will be either positive or negative. In fact, we can summarize this as follows:

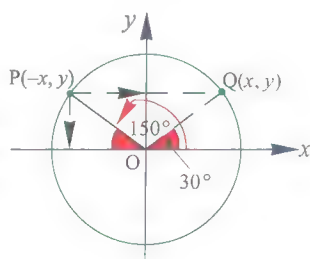


Mnemonic: All Stations To ity

However, knowing the sign of a trigonometric ratio reflects only half the information. We still need to determine the numerical value. We start by considering a few examples.

Consider the value of $\sin 150^\circ$.
Using the unit circle we have:

By symmetry we see that the y -coordinate of Q and the y -coordinate of P are the same and so, $\sin 150^\circ = \sin 30^\circ$.



Therefore, $\sin 150^\circ = \frac{1}{2}$

Note that $150^\circ = 150 \times \frac{\pi}{180} = \frac{5\pi}{6}$ and $30^\circ = \frac{\pi}{6}$,

so that in radian form we have, $\sin \frac{5\pi}{6} = \sin \frac{\pi}{6} = \frac{1}{2}$.

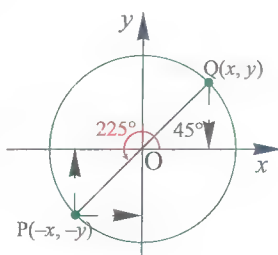
In other words, we were able to express the sine of an angle in the second quadrant in terms of the sine of an angle in the first quadrant. In particular, we have that

$$\text{If } 0^\circ < \theta < 90^\circ, \sin(180^\circ - \theta) = \sin \theta$$

$$\text{If } 0^\circ < \theta < \frac{\pi}{2}, \sin(\pi - \theta) = \sin \theta$$

Next, consider the value of $\cos 225^\circ$.
Using the unit circle we have:

By symmetry we see that the x -coordinate of P has the same magnitude as the x -coordinate of Q but is of the opposite sign.



So, we have that $\cos 225^\circ = -\cos 45^\circ$.

$$\text{Therefore, } \cos 225^\circ = -\frac{1}{\sqrt{2}}.$$

Similarly, as $225^\circ = \frac{3\pi}{4}$ and $45^\circ = \frac{\pi}{4}$,

$$\cos \frac{3\pi}{4} = -\cos \frac{\pi}{4} = -\frac{1}{\sqrt{2}}.$$

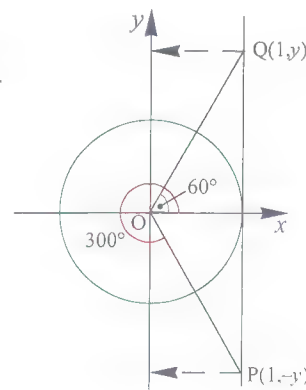
In other words, we were able to express the cosine of an angle in the third quadrant in terms of the cosine of an angle in the first quadrant. In particular, we have that:

$$\text{If } 0^\circ < \theta < 90^\circ, \cos(180^\circ + \theta) = -\cos \theta$$

$$\text{If } 0^\circ < \theta < \frac{\pi}{2}, \cos(\pi + \theta) = -\cos \theta$$

As a last example we consider the value of $\tan 300^\circ$. This time we need to add a tangent to the unit circle cutting the positive x -axis:

By symmetry we see that the y -coordinate of P has the same magnitude as the y -coordinate of Q but is of the opposite sign.



$$\text{So: } \tan 300^\circ = -\tan 60^\circ.$$

$$\text{Therefore, } \tan 300^\circ = -\sqrt{3}.$$

$$\text{As, } 300^\circ = \frac{5\pi}{3}, 60^\circ = \frac{\pi}{3}$$

$$\tan \frac{5\pi}{3} = -\tan \frac{\pi}{3} = -\sqrt{3}.$$

In other words, we were able to express the tangent of an angle in the fourth quadrant in terms of the tangent of an angle in the first quadrant. In particular, we have that:

$\sin(150)$	$\frac{1}{2}$
$\cos(225)$	$-\frac{\sqrt{2}}{2}$
$\tan(300)$	$-\sqrt{3}$

Example C.5.1

Find the exact values of:

a $\cos 120^\circ$

b $\sin 210^\circ$

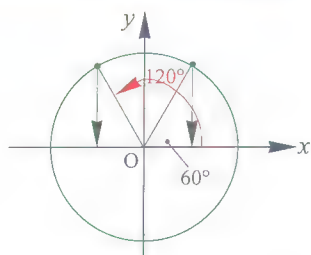
c $\cos \frac{7\pi}{4}$

d $\tan \frac{5\pi}{4}$

a

Step 1: Start by drawing the unit circle:

Step 2: Trace out an angle of 120°



Step 3: Trace out the **reference angle** in the first quadrant. In this case it is 60° .

Step 4: Use the symmetry between the reference angle and the given angle.

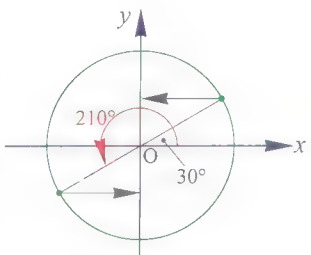
Step 5: State relationship and give answer:

$$\cos 120^\circ = -\cos 60^\circ = -\frac{1}{2}$$

b

Step 1: Start by drawing the unit circle:

Step 2: Trace out an angle of 210°



Step 3: Trace out the **reference angle** in the first quadrant. In this case it is 30° .

Step 4: Use the symmetry between the reference angle and the given angle.

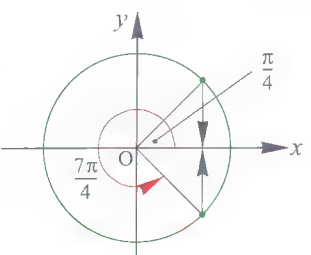
Step 5: State relationship and give answer:

$$\sin 210^\circ = -\sin 30^\circ = -\frac{1}{2}$$

c

Step 1: Start by drawing the unit circle:

Step 2: Trace out an angle of $\frac{7\pi}{4}$ ($= 315^\circ$)



Step 3: Trace out the **reference angle** in the first quadrant. In this case it is $\frac{\pi}{4}$.

Step 4: Use the symmetry between the reference angle and the given angle.

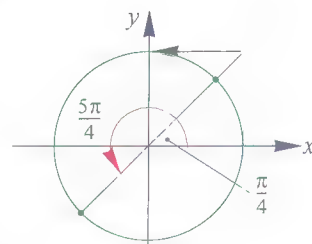
Step 5: State relationship and give answer:

$$\cos \frac{7\pi}{4} = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

d

Step 1: Start by drawing the unit circle:

Step 2: Trace out an angle of $\frac{5\pi}{4}$ ($= 225^\circ$)



Step 3: Trace out the **reference angle** in the first quadrant. In this case it is $\frac{\pi}{4}$.

Step 4: Use the symmetry between the reference angle and the given angle.

Step 5: State relationship and give answer:

$$\tan \frac{5\pi}{4} = \tan \frac{\pi}{4} = 1$$

The results we have obtained, that is, expressing trigonometric ratios of any angle in terms of trigonometric ratios of acute angles in the first quadrant (i.e. reference angles) are known as **trigonometric reduction formulae**. There are too many formulae to commit to memory, and so it is advisable to draw a unit circle and then use symmetry properties as was done in Example C.5.1. We list a number of these formulae in the table below, where $0 < \theta < \frac{\pi}{2}$ ($= 90^\circ$).

Note: From this point on, angles without the degree symbol or radian symbol will mean an angle measured in radian mode.

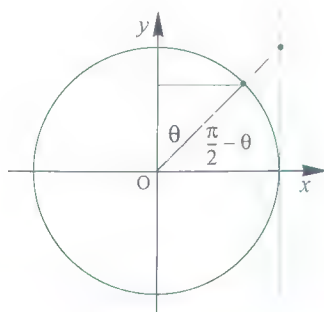
Q	θ in degrees	θ in radians
2	$\sin(180^\circ - \theta) = \sin \theta$	$\sin(\pi - \theta) = \sin \theta$
	$\cos(180^\circ - \theta) = -\cos \theta$	$\cos(\pi - \theta) = -\cos \theta$
	$\tan(180^\circ - \theta) = -\tan \theta$	$\tan(\pi - \theta) = -\tan \theta$
3	$\sin(180^\circ + \theta) = -\sin \theta$	$\sin(\pi + \theta) = -\sin \theta$
	$\cos(180^\circ + \theta) = -\cos \theta$	$\cos(\pi + \theta) = -\cos \theta$
	$\tan(180^\circ + \theta) = \tan \theta$	$\tan(\pi + \theta) = \tan \theta$

Q	θ in degrees	θ in radians
4	$\sin(360^\circ - \theta) = -\sin\theta$	$\sin(2\pi - \theta) = -\sin\theta$
	$\cos(360^\circ - \theta) = \cos\theta$	$\cos(2\pi - \theta) = \cos\theta$
	$\tan(360^\circ - \theta) = -\tan\theta$	$\tan(2\pi - \theta) = -\tan\theta$

There is another set of results that is suggested by symmetry through the fourth quadrant:

Quadrant	θ in degrees or θ in radians
4	$\sin(-\theta) = -\sin\theta$
	$\cos(-\theta) = \cos\theta$
	$\tan(-\theta) = -\tan\theta$

The symmetry of the unit circle leads to a number of further relationships



Quadrant	θ in radians or θ in degrees
1	$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$ $\sin(90^\circ - \theta) = \cos\theta$
	$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$ $\cos(90^\circ - \theta) = \sin\theta$
	$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$ $\tan(90^\circ - \theta) = \cot\theta$

Rather than trying to memorise these results, reference to the unit circle and its symmetries will reveal them when they are needed.

Reciprocals

Note the introduction of a new trigonometric ratio, $\cot\theta$. This is one of a set of three other trigonometric ratios known as the **reciprocal trigonometric ratios**, namely **cosecant**, **secant** and **cotangent** ratios. These are defined as:

cosecant ratio: $\operatorname{cosec}\theta = \frac{1}{\sin\theta}, \sin\theta \neq 0$
 secant ratio: $\sec\theta = \frac{1}{\cos\theta}, \cos\theta \neq 0$
 cotangent ratio: $\cot\theta = \frac{1}{\tan\theta}, \tan\theta \neq 0$

Note then, that $\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$, $\sin\theta \neq 0$ and cosec is often written 'csc'.

Example C.5.2

Find the exact values of:

- a $\sec 45^\circ$ b $\operatorname{cosec} 150^\circ$
 c $\cot \frac{11\pi}{6}$ d $\sec 0$

a $\sec 45^\circ = \frac{1}{\cos 45^\circ} = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)} = \sqrt{2}$

b $\operatorname{cosec} 150^\circ = \frac{1}{\sin 150^\circ} = \frac{1}{\sin 30^\circ} = \frac{1}{\left(\frac{1}{2}\right)} = 2$

c $\cot \frac{11\pi}{6} = \frac{1}{\tan\left(\frac{11\pi}{6}\right)} = \frac{1}{\tan\left(-\frac{\pi}{6}\right)} = \frac{1}{-\tan \frac{\pi}{6}}$
 $= \frac{1}{-\left(\frac{1}{\sqrt{3}}\right)} = -\sqrt{3}$

d $\sec 0 = \frac{1}{\cos 0} = \frac{1}{1} = 1$

Exercise C.5.1

1. Convert the following angles to degrees.

- a $\frac{2\pi}{3}$ b $\frac{3\pi}{5}$
 c $\frac{12\pi}{10}$ d $\frac{5\pi}{18}$

2. Convert the following angles to radians.

- a 180° b 270°
 c 140° d 320°

3. Find the exact value of:

- a $\sin 120^\circ$ b $\cos 120^\circ$
 c $\tan 120^\circ$ d $\sec 120^\circ$
 e $\sin 210^\circ$ f $\cos 210^\circ$

g	$\tan 210^\circ$	h	$\cot 210^\circ$
i	$\sin 225^\circ$	j	$\cos 225^\circ$
k	$\tan 225^\circ$	l	$\operatorname{cosec} 225^\circ$
m	$\sin 315^\circ$	n	$\cos 315^\circ$
o	$\tan 315^\circ$	p	$\sec 315^\circ$
q	$\sin 360^\circ$	r	$\cos 360^\circ$
s	$\tan 360^\circ$	t	$\operatorname{cosec} 360^\circ$

4. Find the exact value of:

a	$\sin \pi$	b	$\cos \pi$
c	$\tan \pi$	d	$\sec \pi$
e	$\sin \frac{3\pi}{4}$	f	$\cos \frac{3\pi}{4}$
g	$\tan \frac{3\pi}{4}$	h	$\operatorname{cosec} \frac{3\pi}{4}$
i	$\sin \frac{7\pi}{6}$	j	$\cos \frac{7\pi}{6}$
k	$\tan \frac{7\pi}{6}$	l	$\cot \frac{7\pi}{6}$
m	$\sin \frac{5\pi}{3}$	n	$\cos \frac{5\pi}{3}$

5. Find the exact value of:

a	$\sin(-210^\circ)$	b	$\cos(-30^\circ)$
c	$\tan(-135^\circ)$	d	$\cos(-420^\circ)$
e	$\cot(-60^\circ)$	f	$\sin(-150^\circ)$
g	$\sec(-135^\circ)$	h	$\operatorname{cosec}(-120^\circ)$

6. Find the exact value of:

a	$\sin\left(\frac{\pi}{6}\right)$	b	$\cos\left(\frac{3\pi}{4}\right)$
c	$\tan\left(-\frac{2\pi}{3}\right)$	d	$\sec\left(-\frac{4\pi}{3}\right)$
e	$\cot\left(-\frac{3\pi}{4}\right)$	f	$\sin\left(-\frac{7\pi}{6}\right)$
g	$\cot\left(-\frac{\pi}{3}\right)$	h	$\cos\left(-\frac{7\pi}{6}\right)$
i	$\operatorname{cosec}\left(\frac{2\pi}{3}\right)$	j	$\tan\left(-\frac{11\pi}{6}\right)$
k	$\sec\left(-\frac{13\pi}{6}\right)$	l	$\sin\left(-\frac{7\pi}{3}\right)$

Extra questions



SL 3.6 The Pythagorean Identity

We have seen a number of important relationships between trigonometric ratios. Relationships that are true for all values of θ are known as **identities**. To signal an identity (as opposed to an equation) the **equivalence** symbol is used, i.e. ' $\equiv$ '.

For example, we can write $(x+1)^2 \equiv x^2 + 2x + 1$ as this statement is true for all values of x . However, we would have to write $(x+1)^2 = x^2 + 1$, as this relationship is only true for some values of x (which need to be determined).

One trigonometric identity is based on the unit circle.

Consider the point $P(x, y)$ on the unit circle,

$$x^2 + y^2 = 1 \quad (1)$$

From the previous section, we know that

$$x = \cos \theta \quad (2)$$

$$y = \sin \theta \quad (3)$$

Substituting (2) and (3) into (1) we have: $(\cos \theta)^2 + (\sin \theta)^2 = 1$ or

$$\sin^2 \theta + \cos^2 \theta = 1 \quad (4)$$

This is known as the fundamental trigonometric identity. Note that we have not used the identity symbol, i.e. we have not written $\sin^2 \theta + \cos^2 \theta \equiv 1$. This is because more often than not, it will be 'obvious' from the setting as to whether a relationship is an identity or an equation. And so, there is a tendency to forgo the formal use of the identity statement and restrict ourselves to the equality statement.

By rearranging the identity we have that $\sin^2 \theta = 1 - \cos^2 \theta$ and $\cos^2 \theta = 1 - \sin^2 \theta$. Similarly we obtain the following two new identities:

Divide both sides of (4) by $\cos^2 \theta$:

$$\frac{\sin^2 \theta + \cos^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta} \Leftrightarrow \frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}$$

$$\Leftrightarrow \tan^2\theta + 1 = \sec^2\theta - (5)$$

Divide both sides of (4) by $\sin^2\theta$:

$$\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \Leftrightarrow \frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\Leftrightarrow 1 + \cot^2\theta = \operatorname{cosec}^2\theta - (6)$$

Example C.5.3

If $\cos\theta = -\frac{3}{5}$, where $\pi \leq \theta \leq \frac{3\pi}{2}$, find:

a $\sin\theta$ b $\tan\theta$

Although we have solved problems like this in by making use of a right-angled triangle, we now solve this question by making use of the trigonometric identities we have just developed.

a From $\sin^2\theta + \cos^2\theta = 1$ we have

$$\sin^2\theta + \left(-\frac{3}{5}\right)^2 = 1 \Leftrightarrow \sin^2\theta + \frac{9}{25} = 1$$

$$\Leftrightarrow \sin^2\theta = \frac{16}{25}$$

$$\therefore \sin\theta = \pm\frac{4}{5}$$

Now, as $\pi \leq \theta \leq \frac{3\pi}{2}$,

this means the angle is in the third quadrant, where the sine value is negative.

Therefore, we have that $\sin\theta = -\frac{4}{5}$.

b Using the identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$,

$$\text{we have } \tan\theta = \frac{(-4/5)}{(-3/5)} = \frac{4}{3}.$$

Example C.5.4

If $\tan\theta = \frac{5}{12}$, where $\pi \leq \theta \leq \frac{3\pi}{2}$, find:

a $\cos\theta$ b $\operatorname{cosec}\theta$

a From the identity $\tan^2\theta + 1 = \sec^2\theta$ we have:

$$\left(\frac{5}{12}\right)^2 + 1 = \sec^2\theta \Leftrightarrow \sec^2\theta = \frac{25}{144} + 1$$

$$\therefore \sec^2\theta = \frac{169}{144}$$

$$\therefore \sec\theta = \pm\frac{13}{12}$$

Therefore, as $\cos\theta = \frac{1}{\sec\theta} \Rightarrow \cos\theta = \pm\frac{12}{13}$.

However, $\pi \leq \theta \leq \frac{3\pi}{2}$, meaning that θ is in the third quadrant.

And so, the cosine is negative. That is, $\cos\theta = -\frac{12}{13}$.

Now, $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$, but,

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \Leftrightarrow \sin\theta = \tan\theta \cos\theta \therefore \sin\theta = \frac{5}{12} \times -\frac{12}{13}$$

$$= -\frac{5}{13}$$

Therefore, $\operatorname{cosec}\theta = \frac{1}{(-5/13)} = -\frac{13}{5}$.

Example C.5.5

Simplify the following expressions.

a $\cos\theta + \tan\theta \sin\theta$ b $\frac{\cos\theta}{1 + \sin\theta} - \frac{1 - \sin\theta}{\cos\theta}$

a $\cos\theta + \tan\theta \sin\theta = \cos\theta + \frac{\sin\theta}{\cos\theta} \sin\theta$

$$= \cos\theta + \frac{\sin^2\theta}{\cos\theta}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\cos\theta}$$

$$= \frac{1}{\cos\theta}$$

$$= \sec\theta$$

$$\begin{aligned}
 \text{b} \quad & \frac{\cos \theta}{1 + \sin \theta} - \frac{1 - \sin \theta}{\cos \theta} \\
 &= \frac{\cos^2 \theta}{(1 + \sin \theta) \cos \theta} - \frac{(1 - \sin \theta)(1 + \sin \theta)}{(1 + \sin \theta) \cos \theta} \\
 &= \frac{\cos^2 \theta}{(1 + \sin \theta) \cos \theta} - \frac{1 - \sin^2 \theta}{(1 + \sin \theta) \cos \theta} \\
 &= \frac{\cos^2 \theta - 1 + \sin^2 \theta}{(1 + \sin \theta) \cos \theta} \\
 &= \frac{(\cos^2 \theta + \sin^2 \theta) - 1}{(1 + \sin \theta) \cos \theta} \\
 &= \frac{1 - 1}{(1 + \sin \theta) \cos \theta} \\
 &= 0
 \end{aligned}$$

Example C.5.6

Show that $\frac{1 - 2\cos^2 \theta}{\sin \theta \cos \theta} = \tan \theta - \cot \theta$.

$$\begin{aligned}
 \text{R.H.S.} &= \tan \theta - \cot \theta \\
 &= \frac{\sin \theta}{\cos \theta} - \frac{\cos \theta}{\sin \theta} \\
 &= \frac{\sin^2 \theta - \cos^2 \theta}{\sin \theta \cos \theta} \\
 &= \frac{(1 - \cos^2 \theta) - \cos^2 \theta}{\sin \theta \cos \theta} \\
 &= \frac{1 - 2\cos^2 \theta}{\sin \theta \cos \theta} \\
 &= \text{L.H.S.}
 \end{aligned}$$

Exercise C.5.2

1. Prove the identity.

$$\text{a} \quad \sin \theta + \cot \theta \cos \theta = \operatorname{cosec} \theta$$

$$\text{b} \quad \frac{\sin \theta}{1 + \cos \theta} + \frac{1 + \cos \theta}{\sin \theta} = 2 \operatorname{cosec} \theta$$

$$\text{c} \quad \frac{\sin^2 \theta}{1 - \cos \theta} = 1 + \cos \theta$$

$$\text{d} \quad 3\cos^2 x - 2 = 1 - 3\sin^2 x$$

$$\text{e} \quad \tan^2 x \cos^2 x + \cot^2 x \sin^2 x = 1$$

$$\text{f} \quad \sec \theta - \sec \theta \sin^2 \theta = \cos \theta$$

$$\text{g} \quad \sin^2 \theta (1 + \cot^2 \theta) - 1 = 0$$

$$\text{h} \quad \frac{1}{1 - \sin \phi} + \frac{1}{1 + \sin \phi} = 2 \sec^2 \phi$$

$$\text{i} \quad \frac{\cos \theta}{1 + \sin \theta} + \tan \theta = \sec \theta$$

$$\text{j} \quad \frac{1 - \sin \theta}{\cos \theta} = \frac{\cos \theta}{1 + \sin \theta}$$

$$\text{k} \quad \frac{1}{\sec x + \tan x} = \sec x - \tan x$$

$$\text{l} \quad \sin x + \frac{\cos^2 x}{1 + \sin x} = 1$$

$$\text{m} \quad \frac{\sec \phi + \operatorname{cosec} \phi}{\tan \phi + \cot \phi} = \sin \phi + \cos \phi$$

$$\text{n} \quad \frac{\sin x + 1}{\cos x} = \frac{\sin x - \cos x + 1}{\sin x + \cos x - 1}$$

$$\text{o} \quad \tan x + \sec x = \frac{\tan x + \sec x - 1}{\tan x - \sec x + 1}$$

2. Prove the following.

$$\text{a} \quad (\sin x + \cos x)^2 + (\sin x - \cos x)^2 = 2$$

$$\text{b} \quad \sec^2 \theta \operatorname{cosec}^2 \theta = \sec^2 \theta + \operatorname{cosec}^2 \theta$$

$$\text{c} \quad \sin^4 x - \cos^4 x = (\sin x + \cos x)(\sin x - \cos x)$$

$$\text{d} \quad \sec^4 x - \sec^2 x = \tan^4 x + \tan^2 x$$

$$\text{e} \quad \frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} = 1 - \sin x \cos x$$

$$\text{f} \quad (\cot x - \operatorname{cosec} x)^2 = \frac{\sec x - 1}{\sec x + 1}$$

$$\text{g} \quad (2b \sin x \cos x)^2 + b^2 (\cos^2 x - \sin^2 x)^2 = b^2$$

3. Eliminate θ from each of the following pairs.

$$\text{a} \quad x = k \sin \theta, y = k \cos \theta$$

$$\text{b} \quad x = b \sin \theta, y = a \cos \theta$$

$$\text{c} \quad x = 1 + \sin \theta, y = 2 - \cos \theta$$

$$\text{d} \quad x = 1 - b \sin \theta, y = 2 + a \cos \theta$$

$$\text{e} \quad x = \sin \theta + 2 \cos \theta, y = \sin \theta - 2 \cos \theta$$

$$4. \quad \text{a} \quad \text{If } \tan \theta = \frac{3}{4}, \pi \leq \theta \leq \frac{3\pi}{2},$$

$$\text{find: i } \cos \theta \quad \text{ii } \operatorname{cosec} \theta$$

$$\text{b} \quad \text{If } \sin \theta = -\frac{3}{4}, \frac{3\pi}{2} \leq \theta \leq 2\pi,$$

$$\text{find: i } \sec \theta \quad \text{ii } \cot \theta$$

5. Solve the following, where $0 \leq \theta \leq 2\pi$:

$$\text{a} \quad 4 \sin \theta = 3 \operatorname{cosec} \theta$$

- b $2\cos^2\theta + \sin\theta - 1 = 0$
 c $2 - \sin\theta = 2\cos^2\theta$
 d $2\sin^2\theta = 2 + 3\cos\theta$

Extra questions



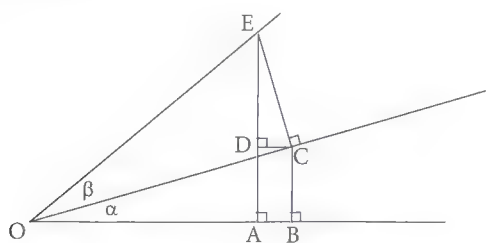
SL 3.6 Compound Angle Identities

As we have seen in the previous section, there are numerous trigonometric identities. However, they were all derived from the fundamental identities. In this section we develop some more fundamental identities (which will lead us to more identities). These fundamental identities are known as compound angle identities. That is, they are identities that involve the sine, cosine and tangent of the sum and difference of two angles.

We start with the sine of the sum of two angles, $\sin(\alpha + \beta)$:

The Diploma Course does not expect students to prove this result. It is included for the sake of completeness.

A commonly given proof of these identities is again only valid for acute angles:



In the figure, $\angle AOE = \alpha + \beta$. The construction lines are drawn with the right angles indicated. Since $\angle DCO = \alpha$ (alternate angles) and $\angle DCE = 90^\circ - \alpha$, it follows that

$$\angle AOE = \alpha + \beta.$$

$$\text{Therefore, we have, } \sin(\alpha + \beta) = \sin AOE = \frac{AE}{OE}$$

$$\begin{aligned} &= \frac{AD + DE}{OE} \\ &= \frac{AD}{OE} + \frac{DE}{OE} \\ &= \frac{BC}{OE} + \frac{DE}{OE} \\ &= \frac{BC}{OC} \times \frac{OC}{OE} + \frac{DE}{EC} \times \frac{EC}{OE} \\ &= \sin\alpha \times \cos\beta + \cos\alpha \times \sin\beta \end{aligned}$$

It is now possible to prove the difference formula, replacing β by $-\beta$ we have:

$$\begin{aligned} \sin(\alpha - \beta) &= \sin(\alpha + (-\beta)) \\ &= \sin\alpha \cos(-\beta) + \cos\alpha \sin(-\beta) \\ &= \sin\alpha \cos\beta - \cos\alpha \sin\beta \end{aligned}$$

$$(\cos(-\beta) = \cos\beta \text{ and } \sin(-\beta) = -\sin\beta)$$

And so we have the addition and difference identities for sine:

$$\begin{aligned} \sin(\alpha + \beta) &= \sin\alpha \cos\beta + \cos\alpha \sin\beta \\ \sin(\alpha - \beta) &= \sin\alpha \cos\beta - \cos\alpha \sin\beta \end{aligned}$$

A similar identity can be derived for the cosine function (using the same diagram):

$$\begin{aligned} \cos(\alpha + \beta) &= \frac{OA}{OE} = \frac{OB - AB}{OE} \\ &= \frac{OB}{OE} - \frac{AB}{OE} \\ &= \frac{OB}{OE} - \frac{CD}{OE} \\ &= \frac{OB}{OC} \times \frac{OC}{OE} - \frac{CD}{EC} \times \frac{EC}{OE} \\ &= \cos\alpha \cos\beta - \sin\alpha \sin\beta \end{aligned}$$

Similarly, from this and replacing β by $-\beta$ we have that $\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$.

And so we have the addition and difference identities for cosine:

$$\begin{aligned} \cos(\alpha + \beta) &= \cos\alpha \cos\beta - \sin\alpha \sin\beta \\ \cos(\alpha - \beta) &= \cos\alpha \cos\beta + \sin\alpha \sin\beta \end{aligned}$$

Also, the tangent addition identity can be proved as follows:

$$\text{Using } \frac{\sin\theta}{\cos\theta} = \tan\theta$$

$$\begin{aligned}\tan(\alpha + \beta) &= \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} \\ &= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} \\ &= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta} \cdot \frac{\cos\alpha\cos\beta + \sin\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} \\ &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}\end{aligned}$$

Again, if we replace β by $-\beta$ we have:

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.$$

And so we have the addition and difference identities for tangent:

$$\begin{aligned}\tan(\alpha + \beta) &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \\ \tan(\alpha - \beta) &= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}\end{aligned}$$

As a special case of the compound identities we have obtained so far, we have a set of identities known as the **double-angle identities**.

Using the substitution $\theta = \alpha = \beta$ we obtain the identities:

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta \\ \cos 2\theta &= \cos^2\theta - \sin^2\theta\end{aligned}$$

i.e. substituting $\theta = \alpha = \beta$ into:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \text{ we obtain}$$

$$\begin{aligned}\sin(\theta + \theta) &= \sin\theta\cos\theta + \cos\theta\sin\theta \\ \therefore \sin 2\theta &= 2\sin\theta\cos\theta\end{aligned}$$

Similarly, substituting $\theta = \alpha = \beta$ into:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \text{ we obtain}$$

$$\begin{aligned}\cos(\theta + \theta) &= \cos\theta\cos\theta - \sin\theta\sin\theta \\ \therefore \cos 2\theta &= \cos^2\theta - \sin^2\theta\end{aligned}$$

The second of these can be further developed to give:

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \cos^2\theta - (1 - \cos^2\theta) = 2\cos^2\theta - 1$$

and

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = (1 - \sin^2\theta) - \sin^2\theta = 1 - 2\sin^2\theta$$

Finally, we have a double-angle identity for the tangent:

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Summary of double-angle identities

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta \\ \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta \\ \tan 2\theta &= \frac{2\tan\theta}{1 - \tan^2\theta}\end{aligned}$$

We have seen how trigonometric identities can be used to solve equations, simplify expressions and to prove further identities. We now illustrate this using the new set of identities.

Example C.5.7

Simplify the expression: $\frac{\sin 3\alpha}{\sin \alpha} - \frac{\cos 3\alpha}{\cos \alpha}$.

$$\begin{aligned}\frac{\sin 3\alpha}{\sin \alpha} - \frac{\cos 3\alpha}{\cos \alpha} &= \frac{\sin 3\alpha\cos \alpha - \cos 3\alpha\sin \alpha}{\sin \alpha\cos \alpha} \\ &= \frac{\sin(3\alpha - \alpha)}{\sin \alpha\cos \alpha} \\ &= \frac{\sin 2\alpha}{\sin \alpha\cos \alpha} \\ &= \frac{1}{2} \cdot \frac{2\sin 2\alpha}{\sin \alpha\cos \alpha} \\ &= 2\end{aligned}$$

Example C.5.8

Prove the identity $\cos 3\alpha = 4\cos^3\alpha - 3\cos \alpha$.

$$\begin{aligned}\cos 3\alpha &= \cos(2\alpha + \alpha) \\ &= \cos 2\alpha\cos \alpha - \sin 2\alpha\sin \alpha \\ &= (2\cos^2\alpha - 1)\cos \alpha - 2\sin \alpha\cos \alpha\sin \alpha \\ &= 2\cos^3\alpha - \cos \alpha - 2\sin^2\alpha\cos \alpha \\ &= 2\cos^3\alpha - \cos \alpha - 2(1 - \cos^2\alpha)\cos \alpha \\ &= 2\cos^3\alpha - \cos \alpha - 2\cos \alpha + 2\cos^3\alpha\end{aligned}$$

$$= 4\cos^3\alpha - 3\cos\alpha$$

Example C.5.9

Using a compound identity, show that $\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin\theta$

$$\begin{aligned}\text{L.H.S.} &= \cos\left(\frac{3\pi}{2} - \theta\right) \\ &= \cos\left(\frac{3\pi}{2}\right)\cos\theta + \sin\left(\frac{3\pi}{2}\right)\sin\theta \\ &= 0 \times \cos\theta + (-1) \times \sin\theta \\ &= -\sin\theta \\ &= \text{R.H.S.}\end{aligned}$$

Example C.5.10

Find the exact value of:

$$\text{a} \quad \cos 15^\circ \quad \text{b} \quad \tan \frac{5\pi}{12}$$

a

$$\cos 15^\circ = \cos(45^\circ - 30^\circ) = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}$$

$$= \frac{\sqrt{3} + 1}{2\sqrt{2}}$$

$$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - 1 \times \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} + 1}{\sqrt{3} - 1}$$

b

$$\tan \frac{5\pi}{12} = \tan\left(\frac{3\pi}{12} + \frac{2\pi}{12}\right) = \frac{\tan \frac{3\pi}{12} + \tan \frac{2\pi}{12}}{1 - \tan \frac{3\pi}{12} \tan \frac{2\pi}{12}} = \frac{\tan \frac{\pi}{4} + \tan \frac{\pi}{6}}{1 - \tan \frac{\pi}{4} \tan \frac{\pi}{6}}$$

Example C.5.11

Prove that $\frac{\sin 2\phi + \sin \phi}{\cos 2\phi + \cos \phi + 1} = \tan \phi$

$$\begin{aligned}\text{L.H.S.} &= \frac{\sin 2\phi + \sin \phi}{\cos 2\phi + \cos \phi + 1} = \frac{2 \sin \phi \cos \phi + \sin \phi}{2 \cos^2 \phi - 1 + \cos \phi + 1} \\ &= \frac{\sin \phi (2 \cos \phi + 1)}{\cos \phi (2 \cos \phi + 1)} \\ &= \frac{\sin \phi}{\cos \phi} \\ &= \tan \phi \\ &= \text{R.H.S.}\end{aligned}$$

Exercise C.5.3

1. Expand the following.

$$\begin{array}{ll}\text{a} & \sin(\alpha + \phi) \quad \text{b} \quad \cos(3\alpha + 2\beta) \\ \text{c} & \sin(2x - y) \quad \text{d} \quad \cos(\phi - 2\alpha) \\ \text{e} & \tan(2\theta - \alpha) \quad \text{f} \quad \tan(\phi - 3\omega)\end{array}$$

2. Simplify the following.

$$\text{a} \quad \sin 2\alpha \cos 3\beta - \sin 3\beta \cos 2\alpha$$

$$\text{b} \quad \cos 2\alpha \cos 5\beta - \sin 2\alpha \sin 5\beta$$

$$\text{c} \quad \sin x \cos 2y + \sin 2y \cos x$$

$$\text{d} \quad \cos x \cos 3y + \sin x \sin 3y$$

$$\text{e} \quad \frac{\tan 2\alpha - \tan \beta}{1 + \tan 2\alpha \tan \beta}$$

$$\text{f} \quad \frac{\tan(x - y) + \tan y}{1 - \tan(x - y) \tan y}$$

$$\text{g} \quad \frac{1 - \tan \phi}{1 + \tan \phi}$$

$$\text{h} \quad \frac{1}{\sqrt{2}} \sin(\alpha + \beta) + \frac{1}{\sqrt{2}} \cos(\alpha + \beta)$$

3. Given that $\sin \theta = \frac{4}{5}$, $0 \leq \theta \leq \frac{\pi}{2}$ and

$$\cos \phi = -\frac{5}{13}, \pi \leq \phi \leq \frac{3\pi}{2}, \text{ evaluate:}$$

- a $\sin(\theta + \phi)$
b $\cos(\theta + \phi)$
c $\tan(\theta - \phi)$

4. Given that $\sin \theta = -\frac{3}{5}$, $\pi \leq \theta \leq \frac{3\pi}{2}$

$$\text{and } \cos \phi = -\frac{12}{13}, \pi \leq \phi \leq \frac{3\pi}{2}, \text{ evaluate:}$$

- a $\sin(\theta - \phi)$
b $\cos(\theta - \phi)$
c $\tan(\theta + \phi)$

5. Given that $\sin \theta = -\frac{5}{6}$, $\frac{3\pi}{2} \leq \theta \leq 2\pi$, evaluate:

- a $\sin 2\theta$
b $\cos 2\theta$
c $\tan 2\theta$
d $\sin 4\theta$

6. Given that $\tan x = -3$, $\frac{\pi}{2} \leq x \leq \pi$, evaluate:

- a $\sin 2x$
b $\cos 2x$
c $\tan 2x$
d $\tan 4x$

7. Find the exact value of:

- a $\sin \frac{5\pi}{12}$ b $\sin 105^\circ$
c $\cos \frac{11\pi}{12}$ d $\tan 165^\circ$

8. Given that $\tan x = \frac{a}{b}$, $\pi \leq x \leq \frac{3\pi}{2}$, evaluate

- a $\sin 2x$ b $\operatorname{cosec} 2x$
c $\cos 4x$ d $\tan 2x$

9. Prove the following identities.

- a $\cot x - \cot 2x = \operatorname{cosec} 2x$
b $\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y$
c $\sec^2 x = 1 + \tan^2 x$
d $\tan(\theta + \phi) + \tan(\theta - \phi) = \frac{2 \sin 2\theta}{\cos 2\theta + \cos 2\phi}$
e $\cos^4 \alpha - \sin^4 \alpha = 1 - 2 \sin^2 \alpha$
f $\frac{1}{\sin y \cos y} - \frac{\cos y}{\sin y} = \tan y$
g $\frac{1 + \cos 2y}{\sin 2y} = \frac{\sin 2y}{1 - \cos 2y}$
h $\csc\left(\theta + \frac{\pi}{2}\right) = \sec \theta$
i $\cos 3x = \cos x - 4 \sin^2 x \cos x$
j $\frac{1 + \sin 2\theta}{\cos 2\theta} = \frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta}$
k $(\cot x + \csc x)^2 = \frac{1 + \cos x}{1 - \cos x}$
l $\sin 3\alpha = 3 \sin \alpha - 4 \sin^3 \alpha$
m $\cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$
n $2 \cot \theta \sin^2 \theta = \sin 2\theta$
o $\tan\left(\frac{\phi}{2}\right) = \csc \phi - \cot \phi$
p $2 \csc x = \tan\left(\frac{x}{2}\right) + \cot\left(\frac{x}{2}\right)$
q $\cos \beta + \sin \beta = \frac{\cos 2\beta}{\cos \beta - \sin \beta}$
r $\tan \alpha + \tan \beta = \frac{\sin(\alpha + \beta)}{\cos \alpha \cos \beta}$

$$s \quad \sin^2 \frac{\theta}{2} = \frac{1 - \cos \theta}{2}$$

$$t \quad \frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} = 1 - \frac{1}{2} \sin 2x$$

Extra examples and questions



Answers



Identities

Identities exist outside the topic of trigonometry.

These examples are due to Giovanni Domenico Cassini (1625 – 1712) and Eugène Charles Catalan (1814 – 1894).

Cassini was an Italian astronomer who was one of the first observers of the rings of Saturn.

Catalan was a Belgian Mathematician.

Their work relates to the Fibonacci sequence:

1	1	2	3	5	8	13	21	34
55	89	144	233	377	610	987	1597	2584

Cassini looked at groups of three consecutive Fibonacci numbers.

The first such group is: 1, 1, 2 which are denoted by:

$$F_1 = 1, F_2 = 1 \text{ \& } F_3 = 2$$

Cassini observed that:

$$F_1 F_3 - F_2^2 = 1 \times 2 - 1 = 1$$

The next such group is: 1, 2, 3 and:

$$F_2 = 1, F_3 = 2 \text{ \& } F_4 = 3$$

$$F_2 F_4 - F_3^2 = 1 \times 3 - 2^2 = -1$$

The next group is: 2, 3, 5 and:

$$F_3 = 2, F_4 = 3 \text{ \& } F_5 = 5$$

$$F_3 F_5 - F_4^2 = 2 \times 5 - 3^2 = -1$$

The next group is: 3, 5, 8 and:

$$F_4 = 3, F_5 = 5 \text{ \& } F_6 = 8$$

$$F_4 F_6 - F_5^2 = 3 \times 8 - 5^2 = -1$$

Cassini proposed that this alternating pattern of -1 and +1 would continue indefinitely and that:

$$F_{n-1} F_{n+1} - F_n^2 = (-1)^n$$

for all n .

Does Cassini's 'identity' work for all n ? Can you prove it? Only then can it correctly be called an identity.

Catalan extended Cassini's ideas to:

$$F_n^2 - F_{n-r} F_{n+r} = (-1)^{n-r} F_r^2$$

For example, if $n = 7$ and $r = 3$:

$$F_7^2 - F_4^2 = 13^2 - 3^2 = 160$$

$$F_7 - F_4 = 13 - 3 = 10$$

$$F_{7+3} = F_{10} = 55$$

So the left hand side is:

$$F_n^2 - F_{n-r} F_{n+r} = 169 - 3 \times 55 = 4$$

On the right hand side:

$$(-1)^{n-r} = (-1)^{7-3} = (-1)^4 = 1$$

$$F_r^2 = F_3^2 = 2^2 = 4$$

$$(-1)^{n-r} F_r^2 = 1 \times 4 = 4$$

So, Catalan's formula works for this example.

But is it an identity?

SYLLABUS NOTE

The study of the Sine Rule and the Ambiguous Case was included in the Section on the Sine Rule in Chapter C2.

This topic can be profitably revisited now that students have additional understanding of the sine function.

C.6 Trigonometric Graphs

SL 3.7

Giant Clam

The sine, cosine and tangent functions

As we saw in Chapter C.5, there is an infinite set of angles all of which give values (when they exist) for the main trigonometric ratios. We also noticed that the trigonometric ratios behave in such a way that values are repeated over and over. This is known as periodic behaviour. Many real world phenomena are periodic in the sense that the same patterns repeat over time. The trigonometric functions are often used to model such phenomena which include sound waves, light waves, alternating current electricity and other more approximately periodic events such as tides and even our moods (i.e. biorhythms).

Notice how we have introduced the term 'trigonometric function', replacing the term 'trigonometric ratio'. By doing this we can extend the use of the trigonometric ratios to a new field of problems.

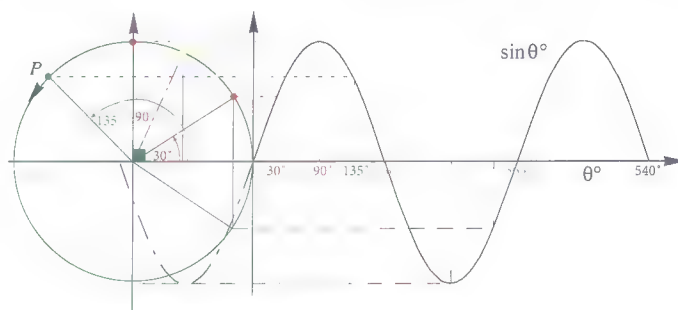
When the trigonometric functions are used for these purposes, the angles are almost always measured in radians. However, there is no reason why we cannot use degrees. It will always be obvious from the equation as to which mode of angle we are using. An expression such as $\sin x$ will imply (by default) that the angle is measured in radians, otherwise it will be written as $\sin x^\circ$, implying that the angle is measured in degrees.

What do trigonometric functions look like?

The sine and cosine values display a periodic nature. This means that, if we were to plot a graph of the sine values versus their angles or the cosine values versus their angles, we could expect their graphs to demonstrate periodic behaviour. We start by plotting points.

The Sine Function

θ	0	30	45	60	90	120	135	150	180	...	330	360
$\sin \theta^\circ$	0.0	0.5	0.71	0.87	1.0	0.86	0.71	0.5	0.0	...	-0.5	0.0



Notice that, as the sine of angle θ corresponds to the y -value of point P on the unit circle, as P moves around the circle in an anticlockwise direction, we can measure the y -value and plot it on a graph as a function of θ (as above).

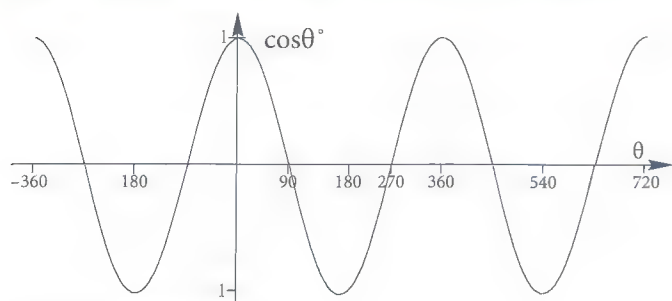
Feature of sine graph:

1. Maximum value = 1, Minimum value = -1
2. Period = 360° (i.e. graph repeats itself every 360°)
3. If P moves in a clockwise direction, y -values continue in their periodic nature (see dashed part of graph).

The Cosine Function

θ	0	30	45	60	90	120	135	150	180	...	330	360
$\cos \theta^\circ$	1.0	0.87	0.71	0.5	0.0	-0.5	-0.7	-0.8	-1.0	...	0.8	1.0

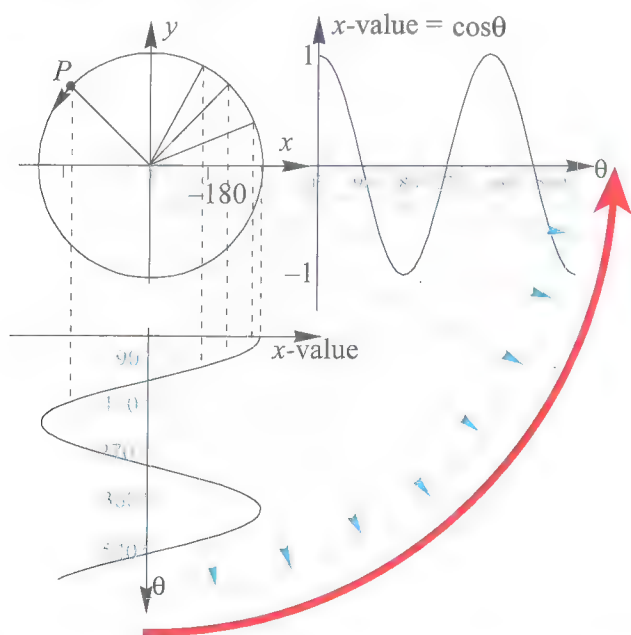
Plotting these points on a $\cos\theta^\circ$ versus θ -axis, we have:



Feature of cosine graph:

1. Maximum value = 1, Minimum value = -1
2. Period = 360° (i.e. graph repeats itself every 360°)
3. If P moves in a clockwise direction, x -values continue in their periodic nature.

There is a note to be made about using the second method (the one used to obtain the sine graph) when dealing with the cosine graph. As the cosine values correspond to the x -values on the unit circle, the actual cosine graph should have been plotted as shown below. However, for the sake of consistency, we convert the 'vertical graph' to the more standard 'horizontal graph':



There are some common observations to be made from these two graphs:

1. We have that the period of each of these functions is 360° . This is the length that it takes for the curve to start repeating itself.
2. The amplitude of the function is the distance between the centre line (in this case the θ -axis) and one of the maximum points. In this case, the amplitude is 1.

The sine and cosine functions are useful for modelling wave phenomena such as sound, light, water waves etc.

The Tangent Function

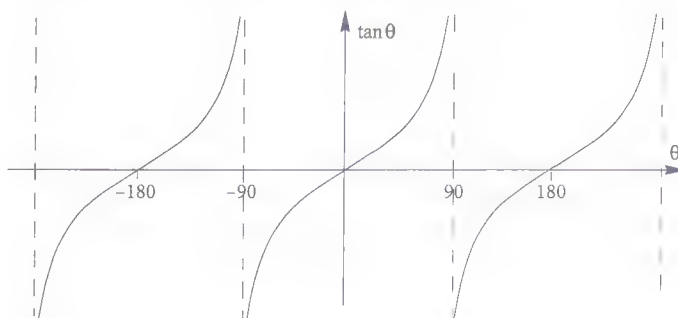
The third trigonometric function (tangent) is defined as:

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

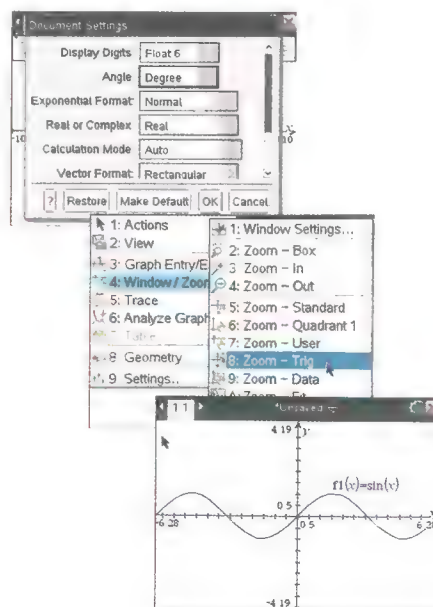
and so is defined for all angles for which the cosine function is non-zero.

The angles for which the tangent function is not defined correspond to the x -axis intercepts of the cosine function which are $\pm 90^\circ, \pm 270^\circ, \pm 450^\circ, \dots$. At these points the graph of the tangent function has vertical asymptotes.

The period of the tangent function is 180° , which is half that of the sine and cosine functions. Since the tangent function has a vertical asymptote, it cannot be said to have an amplitude. It is also generally true that the tangent function is less useful than the sine and cosine functions for modelling applications. The graph of the basic tangent function is:



When sketching these graphs using a calculator, be sure that the **WINDOW** settings are appropriate for the **MODE** setting. In the case of degrees we have:



Transformations of Trigonometric Functions

We now consider some of the possible transformations that can be applied to the standard sine and cosine functions and look at how these transformations affect the basic properties of both these graphs.

1 Vertical translations

Functions of the type $f(x) = \sin(x) + c$, $f(x) = \cos(x) + c$ and $f(x) = \tan(x) + c$ represent vertical translations of the curves of $\sin(x)$, $\cos(x)$ and $\tan(x)$ respectively. If $c > 0$ the graph is moved vertically up and if $c < 0$ the graph is moved vertically down.

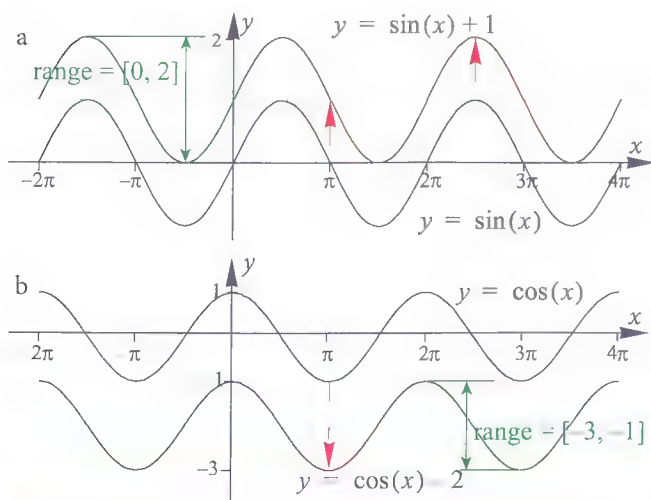
Example C.6.1

Sketch the graphs of the following functions for x -values in the range -2π to 4π .

a $y = \sin(x) + 1$ b $y = \cos(x) - 2$

A graph sketch should show all the important features of a graph. In this case, the axes scales are important and should show the correct period (2π) and ranges a: $[0, 1]$, b: $[-3, -1]$.

That is, adding or subtracting a fixed amount to a trigonometric function translates the graph parallel to the y -axis.



2 Horizontal translations

Functions of the type $f(x) = \sin(x \pm \alpha)$, $f(x) = \cos(x \pm \alpha)$ and $f(x) = \tan(x \pm \alpha)$ where $\alpha > 0$ are horizontal translations of the curves $\sin(x)$, $\cos(x)$ and $\tan(x)$ respectively. In the context of trigonometric functions, horizontal translations are often referred to as **phase shifts**.

So that:

$$f(x) = \sin(x - \alpha), f(x) = \cos(x - \alpha) \text{ and } f(x) = \tan(x - \alpha)$$

are translations to the right.

while

$$f(x) = \sin(x + \alpha), f(x) = \cos(x + \alpha) \text{ and } f(x) = \tan(x + \alpha)$$

are translations to the left.

Example C.6.2

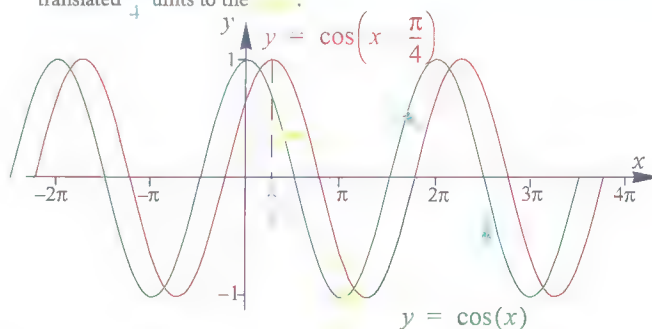
For $-2\pi \leq x \leq 4\pi$, sketch the graphs of the curves with equations:

a $y = \cos\left(x - \frac{\pi}{4}\right)$ b $y = \cos\left(x + \frac{\pi}{3}\right)$
 c $y = \tan\left(x - \frac{\pi}{6}\right)$

a

This is the basic cosine graph

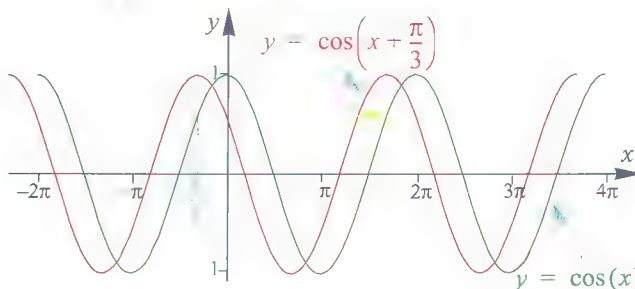
translated $\frac{\pi}{4}$ units to the right.



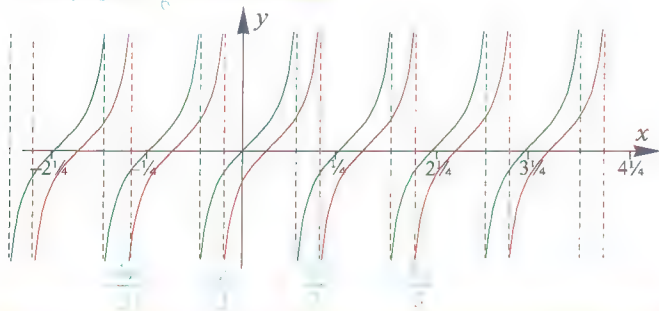
b

This graph is the basic cosine graph

translated $\frac{\pi}{3}$ units to the left.



This is the tangent graph translated $\frac{\pi}{6}$ units to the $\frac{\pi}{6}$. This also translates the asymptotes $\frac{\pi}{6}$ units to the $\frac{\pi}{6}$.

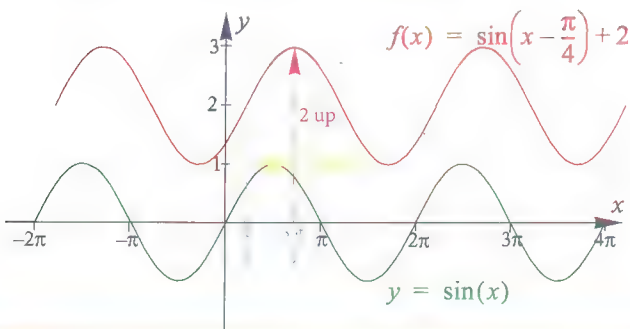


Of course, it is also possible to combine vertical and horizontal translations, as the next example shows.

Example C.6.3

Sketch the graph of the function $f(x) = \sin\left(x - \frac{\pi}{4}\right) + 2$ $-2\pi \leq x \leq 4\pi$.

Basic sine graph translated $\frac{\pi}{4}$ units right and 2 up.



3

Dilations

Functions of the form $f(x) = a \sin(x)$, $f(x) = a \cos(x)$ and $f(x) = a \tan(x)$ are dilations of the curves $\sin(x)$, $\cos(x)$ and $\tan(x)$ respectively, parallel to the y -axis.

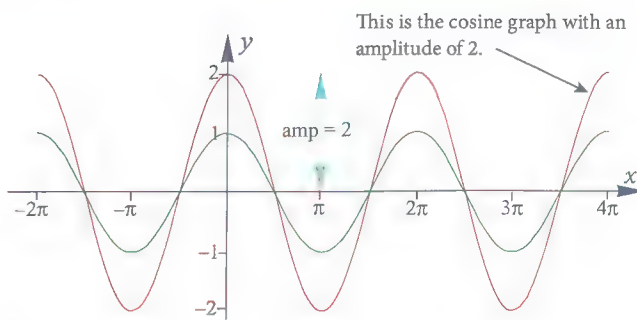
In the case of the sine and cosine functions, the amplitude becomes $|a|$ and not 1. This dilation does not affect the shape of the graph. Also, as the tangent function extends indefinitely, the term amplitude has no relevance.

Example C.6.4

Sketch the graphs of the following functions for $-2\pi \leq x \leq 4\pi$.

a $f(x) = 2 \cos x$ b $f(x) = \frac{1}{3} \sin x$

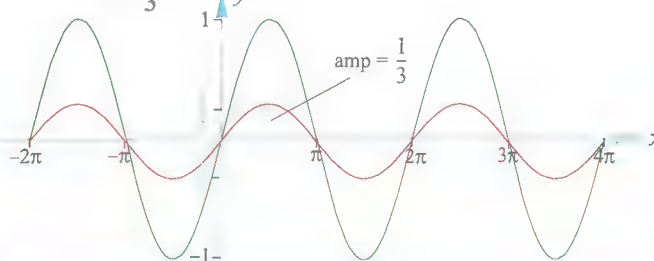
a



b

This is the sine graph with an

amplitude of $\frac{1}{3}$.



Functions of the form $f(x) = \sin(bx)$, $f(x) = \cos(bx)$ and $f(x) = \tan(bx)$ are dilations of the curves $\sin(x)$, $\cos(x)$ and $\tan(x)$ respectively, parallel to the x -axis.

This means that the period of the graph is altered. It can be valuable to remember and use the formula that relates the value of b to the period τ of the dilated function:

1. The graph of $f(x) = \cos(bx)$ will show b cycles in 2π radians, meaning that its period will be given by $\tau = \frac{2\pi}{b}$.
2. The graph of $f(x) = \sin(bx)$ will show b cycles in 2π radians, meaning that its period will be given by $\tau = \frac{2\pi}{b}$.
3. The graph of $f(x) = \tan(bx)$ will show b cycles in π radians, meaning that its period will be given by $\tau = \frac{\pi}{b}$.

Note: In the case of the tangent function, whose original period is π , the new period is $\tau = \frac{\pi}{b}$.

Example C.6.5

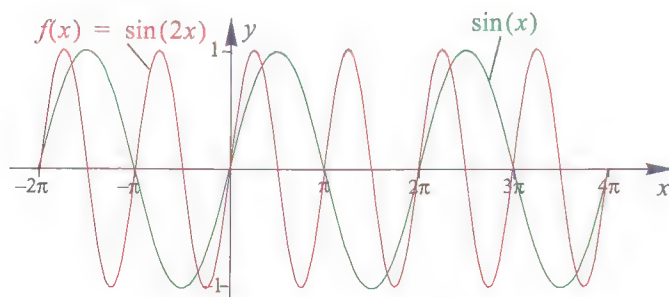
Sketch graphs of the following functions for x -values in the range -2π to 4π .

a $f(x) = \sin(2x)$ b $f(x) = \cos\left(\frac{x}{2}\right)$

c $f(x) = \tan\left(\frac{x}{4}\right)$

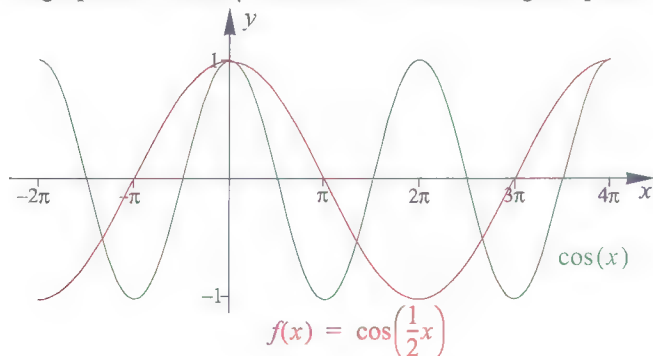
a The value of n is 2 so the period is $\tau = \frac{2\pi}{n} = \frac{2\pi}{2} = \pi$.

Note that this means that the period is **half** that of the basic sine function.

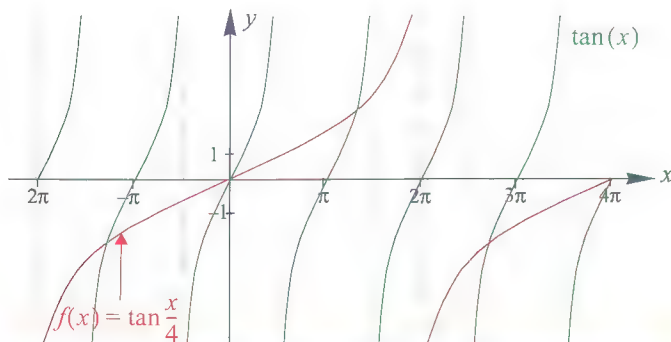


b In this case the value of $n = \frac{1}{2}$ and the period $\tau = \frac{2\pi}{n} = \frac{2\pi}{1/2} = 4\pi$.

The graph is effectively stretched to twice its original period.



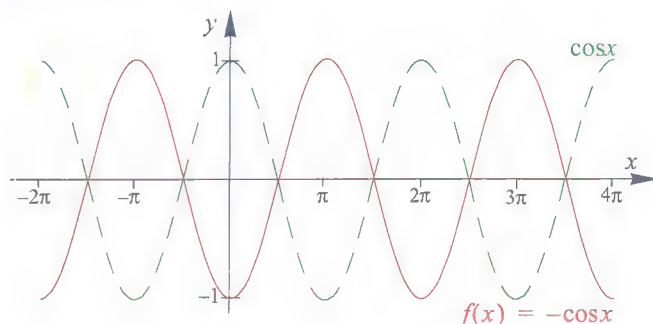
c In this case, with $f(x) = \tan\left(\frac{x}{4}\right)$, the value of $n = \frac{1}{4}$ and the period $\tau = \frac{\pi}{n} = 4\pi$.



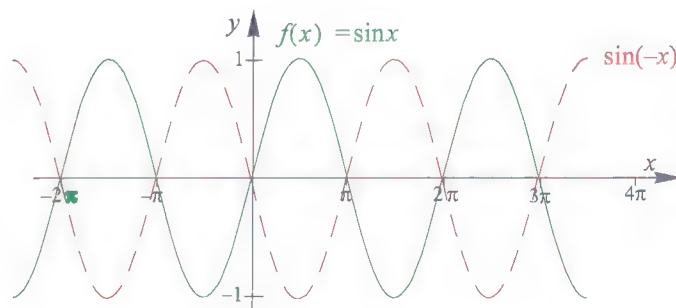
4 Reflections

Recall that the graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected about the x -axis, while that of $y = f(-x)$ is the graph of $y = -f(x)$ reflected about the y -axis.

$f(x) = -\cos x$ is the basic cosine graph (green broken line) reflected in the x -axis.



$f(x) = \sin(-x)$ is the basic sine graph reflected in the y -axis.



5 Combined transformations

You may be required to combine some or all of these transformations in a single function.

The functions of the type:

$$f(x) = a \sin[b(x+c)] + d \quad \text{and} \quad f(x) = a \cos[b(x+c)] + d$$

have:

1. an amplitude of $|a|$ (i.e. the absolute value of a).
2. a period of $\frac{2\pi}{b}$
3. a horizontal translation of c units, $c > 0 \Rightarrow$ to the left
 $c < 0 \Rightarrow$ to the right.

4. a vertical translation of d units, $d > 0 \Rightarrow$ up $d < 0 \Rightarrow$ down.

Care must be taken with the horizontal translation.

For example, the function $f(x) = 2\cos\left(3x + \frac{\pi}{2}\right) - 1$

has a horizontal translation of $\frac{\pi}{6}$ to the left, not $\frac{\pi}{2}$!

This is because:

$$f(x) = 2\cos\left(3x + \frac{\pi}{2}\right) - 1 = 2\cos\left[3\left(x + \frac{\pi}{6}\right)\right] - 1.$$

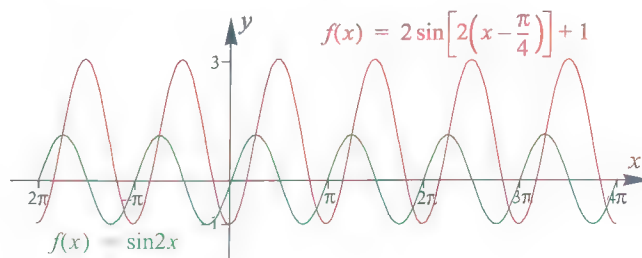
i.e. if the coefficient of x is not one, we must first express the function in the form $a\cos[b(x+c)]+d$.

Similarly we have: $f(x) = a\tan[b(x+c)]+d$

1. no amplitude (as it is not appropriate for the tan function).
2. a period of $\frac{\pi}{b}$
3. a horizontal translation of c units, $c > 0 \Rightarrow$ to the left $c < 0 \Rightarrow$ to the right.
4. a vertical translation of d units, $d > 0 \Rightarrow$ up $d < 0 \Rightarrow$ down.

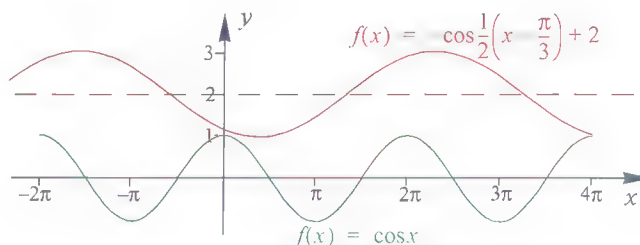
a $f(x) = 2\sin\left[2\left(x - \frac{\pi}{4}\right)\right] + 1.$

This graph has an amplitude of 2, a period of π , a horizontal translation of $\frac{\pi}{4}$ units to the right and a vertical translation of 1 unit up.



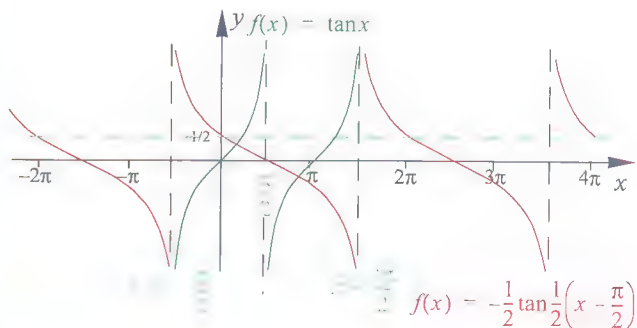
b $f(x) = -\cos\frac{1}{2}\left(x - \frac{\pi}{3}\right) + 2$

The transformations are a reflection in the x -axis, a dilation of factor 2 parallel to the x -axis and a translation of $\frac{\pi}{3}$ right and 2 up.



c $f(x) = -\frac{1}{2}\tan\frac{1}{2}\left(x - \frac{\pi}{2}\right).$

The transformations are a reflection in the x -axis, a vertical dilation with factor $\frac{1}{2}$, a horizontal dilation with factor 2 and a translation of $\frac{\pi}{2}$ to the right.



Example C.6.6

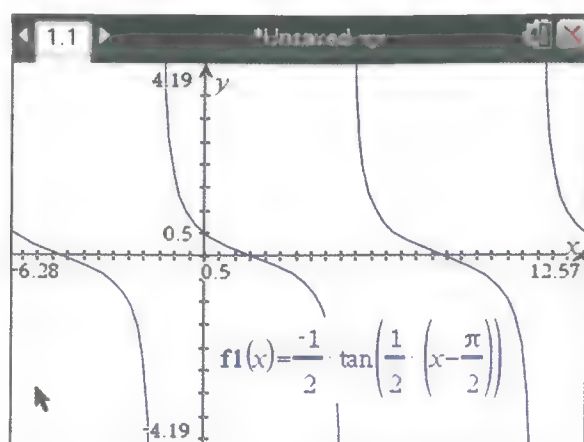
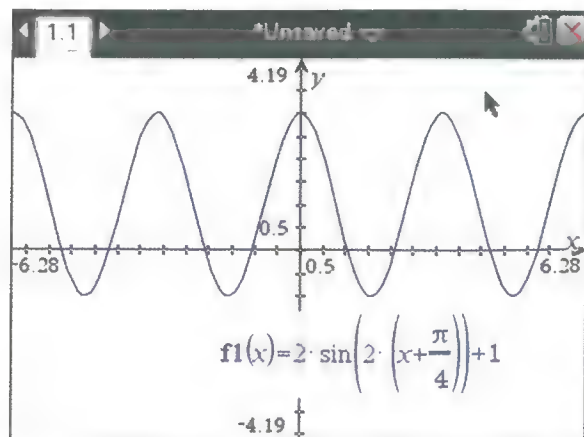
Sketch graphs of the following functions for x -values in the range -2π to 4π .

a $f(x) = 2\sin\left[2\left(x - \frac{\pi}{4}\right)\right] + 1$

b $f(x) = -\cos\frac{1}{2}\left(x - \frac{\pi}{3}\right) + 2$

c $f(x) = -\frac{1}{2}\tan\frac{1}{2}\left(x - \frac{\pi}{2}\right)$

Again we see that a graphics calculator is very useful in such situations – in particular it allows for a checking process.



Exercise C.6.1

1. State the period of the following functions.

a $f(x) = \sin \frac{1}{2}x$

b $f(x) = \cos 3x$

c $f(x) = \tan \frac{x}{3}$

d $g(x) = \cos\left(\frac{x}{2} - \pi\right)$

e $g(x) = 4 \sin(\pi x + 2)$

f $g(x) = 3 \tan\left(\frac{\pi}{2} - 2x\right)$

2. State the amplitude of the following functions.

a $f(x) = 5 \sin 2x$

b $g(x) = -3 \cos \frac{x}{2}$

c $g(x) = 4 - 5 \cos(2x)$

d $f(x) = \frac{1}{2} \sin(3x)$

3. Find the period and, where appropriate, the amplitude of the following functions.

a $y = 2 \sin x$

b $y = 3 \cos \frac{x}{3}$

c $y = 3 \tan x$

d $2 \tan(x - 2\pi)$

e $y = -4 \sin\left[2\left(x + \frac{\pi}{6}\right)\right] + 1$

f $y = 2 - 3 \cos(2x - \pi)$

g $y = -2 \tan \frac{x}{6}$

h $y = \frac{1}{4} \cos\left[3\left(x - \frac{3\pi}{4}\right)\right] + 5$

i $y = 4 \tan\left(\frac{x-4}{3}\right) - 3$

j $y = \frac{2}{3} \cos\left(\frac{3}{4}\left(x + \frac{3\pi}{5}\right)\right) + 5$

4. Sketch the graphs of the curves with equations:

a $y = 3 \cos x, 0 \leq x \leq 2\pi$

b $y = \sin \frac{x}{2}, -\pi \leq x \leq \pi$

c $y = 2 \cos\left(\frac{x}{3}\right), 0 \leq x \leq 3\pi$

d $y = -\frac{1}{2}\sin 3x, 0 \leq x \leq \pi$

e $y = 4\tan\left(\frac{x}{2}\right), 0 \leq x \leq 2\pi$

f $y = \tan(-2x), -\pi \leq x \leq \frac{\pi}{4}$

g $y = \frac{1}{3}\cos(-3x), -\frac{\pi}{3} \leq x \leq \frac{\pi}{3}$

h $y = 3\sin(-2x), -\pi \leq 0 \leq \pi$

5. Sketch the graphs of the curves with equations:

a $y = 3\cos x + 3, 0 \leq x \leq 2\pi$

b $y = \sin\frac{x}{2} - 1, -\pi \leq x \leq \pi$

c $y = 2\cos\left(\frac{x}{3}\right) - 2, 0 \leq x \leq 3\pi$

d $y = -\frac{1}{2}\sin 3x + 2, 0 \leq x \leq \pi$

e $y = 4\tan\left(\frac{x}{2}\right) - 1, 0 \leq x \leq 2\pi$

f $y = \tan(-2x) + 2, -\pi \leq x \leq \frac{\pi}{4}$

g $y = \frac{1}{3}\cos(-3x) + \frac{1}{3}, -\frac{\pi}{3} \leq x \leq \frac{\pi}{3}$

h $y = 3\sin(-2x) - 2, -\pi \leq 0 \leq \pi$

6. Sketch the graphs of the curves with equations:

a $y = 3\cos\left(x + \frac{\pi}{2}\right), 0 \leq x \leq 2\pi$

b $y = \sin\left(\frac{x}{2} - \pi\right), -\pi \leq x \leq \pi$

c $y = 2\cos\left(\frac{x}{3} + \frac{\pi}{6}\right), 0 \leq x \leq 3\pi$

d $y = -\frac{1}{2}\sin(3x + 3\pi), 0 \leq x \leq \pi$

e $y = 4\tan\left(\frac{x}{2} - \frac{\pi}{4}\right), 0 \leq x \leq 2\pi$

f $y = \tan(-2x + \pi), -\pi \leq x \leq \frac{\pi}{4}$

g $y = \frac{1}{3}\cos(-3x - \pi), -\frac{\pi}{3} \leq x \leq \frac{\pi}{3}$

h $y = 3\sin\left(-2x - \frac{\pi}{2}\right), -\pi \leq 0 \leq \pi$

Extra questions



Answers



C.7 Trigonometric Equations

SL 3.8

This chapter will deal with a few of the techniques that are used to solve equations that involve trigonometric functions. There are two basic methods that can be used when solving trigonometric equations:

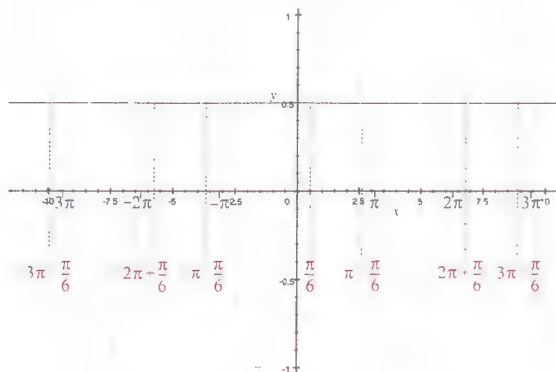
Method 1. Use the unit circle as a visual aid.

Method 2. Use the graph of the function as a visual aid.

The method you choose depends entirely on what you feel comfortable with. However, it is recommended that you become familiar with both methods.

Solution of $\sin x = a$, $\cos x = a$ and $\tan x = a$

The equation $\sin x = \frac{1}{2}$ produces an infinite number of solutions. This can be seen from the graph of the sine function.



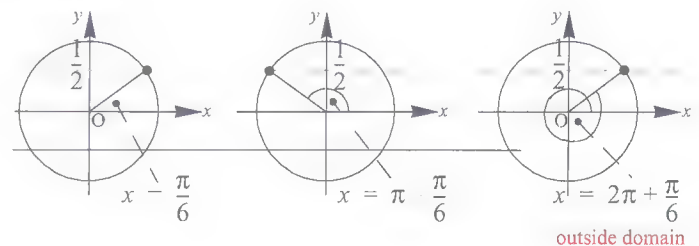
Using the principal angle $\left(\frac{\pi}{6}\right)$ and symmetry, the solutions generated are:

$$\begin{aligned} \text{For } x \geq 0 \quad x &= \frac{\pi}{6}, \pi - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}, 3\pi - \frac{\pi}{6}, \dots \\ &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \dots \end{aligned}$$

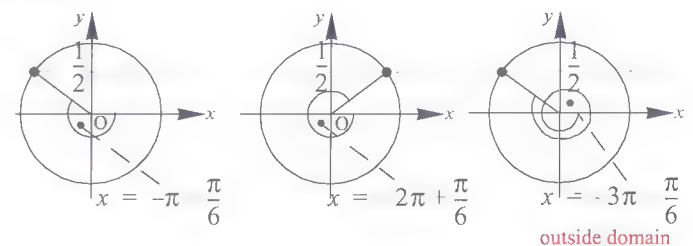
$$\begin{aligned} \text{For } x < 0 \quad x &= -\pi - \frac{\pi}{6}, -2\pi + \frac{\pi}{6}, -3\pi - \frac{\pi}{6}, \dots \\ &= -\frac{7\pi}{6}, -\frac{11\pi}{6}, -\frac{19\pi}{6}, \dots \end{aligned}$$

The same problem could have been solved using Method 1. We start by drawing a unit circle and we continue to move around the circle until we have all the required solutions within the domain restriction. Again, the use of symmetry plays an important role in solving these equations.

For $x \geq 0$:



For $x < 0$:



Again, for the restricted domain $-2\pi < x < \pi$, we have:

$$\sin x = \frac{1}{2} \text{ if } x = -\frac{7\pi}{6}, -\frac{11\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}.$$

The process is identical for the cosine and tangent functions.

Example C.7.1

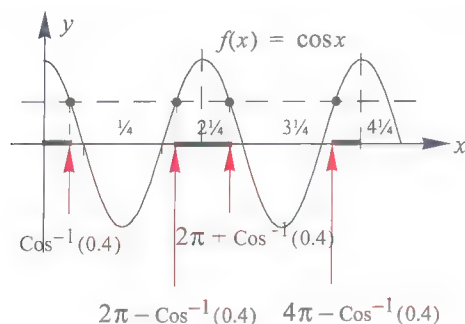
Solve the following, for $0 \leq x \leq 4\pi$, giving answers to 4 decimal places if no exact answers are available.

- a $\cos x = 0.4$ b $\tan x = -1$
c $5 \cos x - 2 = 0$

a Step 1: Find the reference angle:

$$x = \cos^{-1}(0.4) = 1.1593.$$

Step 2: Sketch the cosine graph (or use the unit circle):



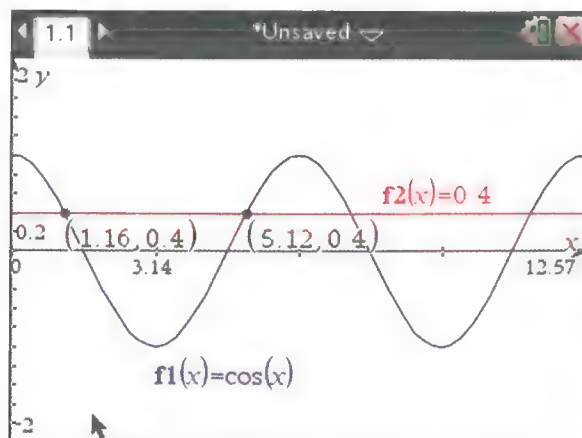
Step 3: Use the reference angle and symmetry to obtain solutions.

Therefore, solutions are,

$$\begin{aligned} &\cos^{-1}(0.4), 2\pi - \cos^{-1}(0.4), 2\pi + \cos^{-1}(0.4), 4\pi - \cos^{-1}(0.4) \\ &= 1.1593, 5.1239, 7.4425, 11.4071 \end{aligned}$$

Step 4: Check that:

- all solutions are within the domain
- you have obtained all the solutions in the domain.

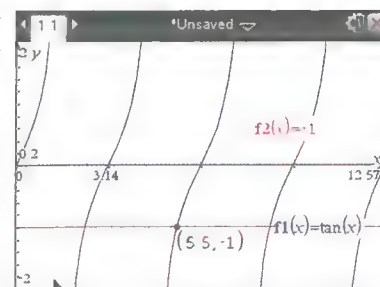


Step1: Find the reference angle (in first quadrant):

$$\tan^{-1}(1) = \frac{\pi}{4} = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}$$

From a calculator, we see that there are four solutions.

Then, as the four solutions obtained all lie in the interval $[0, 4\pi]$, Step 4 is satisfied.



$$\begin{aligned} \text{c } 5 \cos x - 2 &= 0 \Leftrightarrow 5 \cos x = 2 \Leftrightarrow \cos x = \frac{2}{5} \\ \text{i.e. } \cos x &= 0.4 \end{aligned}$$

This is, in fact, identical to the equation in part a and so, we have that $x = 1.1593, 5.1239, 7.4425, 11.4071$

Part c Example C.7.1 highlights the fact that it is possible to transpose a trigonometric equation into a simpler form, which can then be solved. Rather than remembering (or trying to commit to memory) the different possible forms of trigonometric equations and their specific solution processes, the four steps used (with possibly some algebraic manipulation) will always transform a (seemingly) difficult equation into one having a simpler form, as in Example C.7.1.

Some forms of trigonometric equations are:

$$\sin(kx) = a, \cos(x+c) = a, \tan(kx+c) = a,$$

$$b \cos(kx+c) = a, b \sin(kx+c) + d = a$$

And, of course, then there are equations involving the secant, cosecant and cotangent functions.

However, even the most involved of these equations, e.g. $b \sin(kx+c) + d = a$, can be reduced to a simpler form:

1. Transpose:

$$\begin{aligned} b \sin(kx+c) + d &= a \Leftrightarrow b \sin(kx+c) = a-d \\ \Leftrightarrow \sin(kx+c) &= \frac{a-d}{b} \end{aligned}$$

2. Substitute:

Then, setting $kx+c = \theta$ and $\frac{a-d}{b} = m$, we have $\sin \theta = m$ which can be readily solved.

3. Solve for new variable:

So that the solutions to $\sin \theta = m$ are $\theta = \theta_1, \theta_2, \theta_3, \dots$

4. Solve for original variable:

We substitute back for θ and solve for x :

$$kx + c = \theta_1, \theta_2, \theta_3, \dots$$

$$\Leftrightarrow kx = \theta_1 - c, \theta_2 - c, \theta_3 - c, \dots$$

$$\Leftrightarrow x = \frac{\theta_1 - c}{k}, \frac{\theta_2 - c}{k}, \frac{\theta_3 - c}{k}, \dots$$

All that remains is to check that all the solutions have been obtained and that they all lie in the restricted domain.

The best way to see how this works is through some examples.

Example C.7.2

Solve the following, for $0 \leq x \leq 2\pi$.

a $\cos(2x) = 0.4$ b $\tan\left(\frac{1}{2}x\right) = -1$

a Let $\theta = 2x$, so that we now solve the equation $\cos \theta = 0.4$.

From Example C.7.1 a we already have the solutions, namely;

$$\cos^{-1}(0.4), 2\pi - \cos^{-1}(0.4), 2\pi + \cos^{-1}(0.4), 4\pi - \cos^{-1}(0.4)$$

$$= 1.1593, 5.1239, 7.4425, 11.4071$$

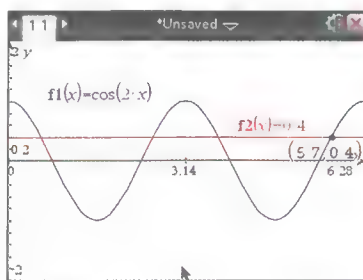
However, we want to solve for x not θ . So, we substitute back for x :

$$\text{i.e. } 2x = 1.1593, 5.1239, 7.4425, 11.4071$$

$$\therefore x = 0.5796, 2.5620, 3.7212, 5.7045$$

To check that we have all the solutions, we sketch the graphs of $y = \cos(2x)$ and $y = 0.4$ over the domain $0 \leq x \leq 2\pi$.

The diagram shows that there should be four solutions.



This time, to solve $\tan\left(\frac{1}{2}x\right) = -1$ we first let $\theta = \frac{1}{2}x$ so that

we now need to solve the simpler equation $\tan \theta = -1$.

b From Example C.7.1 b we have that

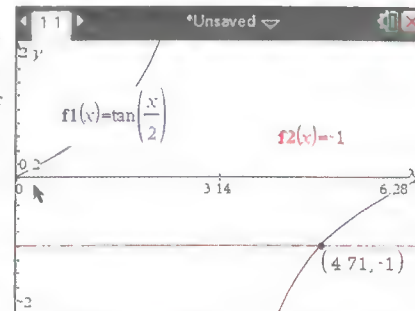
$$\begin{aligned} \theta &= \pi - \frac{\pi}{4}, 2\pi - \frac{\pi}{4}, 3\pi - \frac{\pi}{4}, 4\pi - \frac{\pi}{4} \\ &= \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4} \end{aligned}$$

However, we want to solve for x , not θ . So, we substitute for x :

$$\begin{aligned} \frac{1}{2}x &= \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4} \\ \therefore x &= \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, \frac{15\pi}{2} \end{aligned}$$

To check that we have all the solutions, we sketch the graphs of $y = \tan\left(\frac{1}{2}x\right)$ and $y = -1$ over the domain $0 \leq x \leq 2\pi$.

The diagram shows that there should be only one solution.



Therefore, the only solution is $x = \frac{9\pi}{2}$.

There is of course another step that could be used to help us predetermine which solutions are valid. This requires that we make a substitution not only into the equation, but also into the restricted domain statement.

In Example C.7.2 b, after setting $\theta = \frac{1}{2}x$ to give $\tan \theta = -1$,

we next adjust the restricted domain: $\theta = \frac{1}{2}x \Leftrightarrow x = 2\theta$

So, from $0 \leq x \leq 2\pi$ we now have $0 \leq 2\theta \leq 2\pi \Leftrightarrow 0 \leq \theta \leq \pi$

That is, we have the equivalent equations:

$$\tan\left(\frac{1}{2}x\right) = -1, 0 \leq x \leq 2\pi; \tan \theta = -1, 0 \leq \theta \leq \pi$$

Example C.7.3

Solve: $4\sin(3x) = 2$, $0^\circ \leq x \leq 360^\circ$.

$$4\sin(3x) = 2 \Leftrightarrow \sin(3x) = 0.5, 0^\circ \leq x \leq 360^\circ.$$

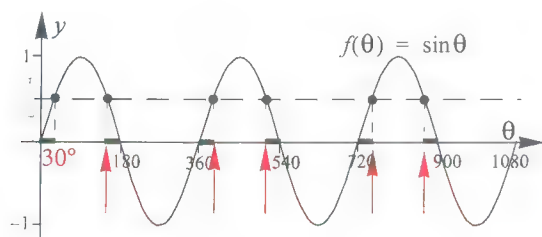
$$\text{Let } \theta = 3x \Rightarrow \sin \theta = 0.5.$$

New domain:

$$0^\circ \leq x \leq 360^\circ \Leftrightarrow 0^\circ \leq \frac{\theta}{3} \leq 360^\circ \Leftrightarrow 0^\circ \leq \theta \leq 1080^\circ$$

Therefore, we have, $\sin \theta = 0.5$, $0^\circ \leq \theta \leq 1080^\circ$.

The reference angle is 30° . Then, by symmetry, we have:



$$\therefore \theta = 30^\circ, 180^\circ - 30^\circ, 360^\circ + 30^\circ, 540^\circ - 30^\circ, \\ 720^\circ + 30^\circ, 900^\circ - 30^\circ$$

$$\therefore 3x = 30^\circ, 150^\circ, 390^\circ, 510^\circ, 750^\circ, 870^\circ$$

$$\therefore x = 10^\circ, 50^\circ, 130^\circ, 170^\circ, 250^\circ, 290^\circ$$

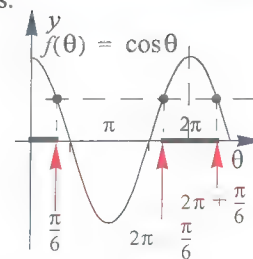
All solutions lie within the **original** specified domain, $0^\circ \leq x \leq 360^\circ$.

Therefore, our equivalent statement is.

$$\cos \theta = \frac{\sqrt{3}}{2}, 0 \leq \theta \leq \frac{5\pi}{2}$$

$$\therefore \theta = \frac{\pi}{6}, 2\pi - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}$$

$$= \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}$$



But we still need to find the x -values:

Therefore, substituting $\theta = \frac{x}{2} + \frac{\pi}{2}$ back into the solution set, we have:

$$\frac{x}{2} + \frac{\pi}{2} = \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}$$

$$\Leftrightarrow x + \pi = \frac{\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}$$

$$\Leftrightarrow x = -\frac{2\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}$$

And, we notice that all solutions lie within the **original** specified domain, $-\pi \leq x \leq 4\pi$.

Example C.7.5

Solve the following, for $0 \leq x \leq 2\pi$.

a $2\sin x = 3\cos x$

b $2\sin 2x = 3\cos x$

c $\sin x = \cos 2x$

$$a \quad 2\sin x = 3\cos x \Leftrightarrow \frac{\sin x}{\cos x} = \frac{3}{2}, 0 \leq x \leq 2\pi$$

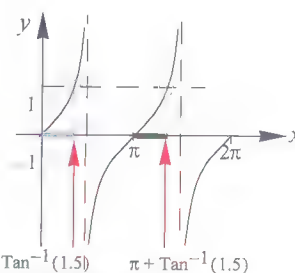
$$\Leftrightarrow \tan x = 1.5, 0 \leq x \leq 2\pi$$

The reference angle is

$$\tan^{-1}(1.5) \approx 0.9828$$

$$x = \tan^{-1}(1.5), \pi + \tan^{-1}(1.5)$$

$$\approx 0.9828, 4.1244$$



b In this case we make use of the double-angle identity, $\sin 2x = 2\sin x \cos x$.

$$\therefore 2\sin 2x = 3\cos x \Leftrightarrow 2(2\sin x \cos x) = 3\cos x$$

$$\Leftrightarrow 4\sin x \cos x - 3\cos x = 0$$

$$\Leftrightarrow \cos x(4\sin x - 3) = 0$$

$$\cos x = 0, \sin x = \frac{3}{4}$$

Example C.7.4

Solve $2\cos\left(\frac{x}{2} + \frac{\pi}{2}\right) - \sqrt{3} = 0$, for $-\pi \leq x \leq 4\pi$.

$$2\cos\left(\frac{x}{2} + \frac{\pi}{2}\right) - \sqrt{3} = 0 \Leftrightarrow \cos\left(\frac{x}{2} + \frac{\pi}{2}\right) = \frac{\sqrt{3}}{2}, -\pi \leq x \leq 4\pi.$$

$$\text{Let } \frac{x}{2} + \frac{\pi}{2} = \theta \Rightarrow \cos \theta = \frac{\sqrt{3}}{2}$$

$$\text{Next, } \frac{x}{2} + \frac{\pi}{2} = \theta \Leftrightarrow \frac{x}{2} = \theta - \frac{\pi}{2} \Leftrightarrow x = 2\theta - \pi.$$

(Obtain x in terms of θ)

New domain:

$$-\pi \leq x \leq 4\pi \Leftrightarrow -\pi \leq 2\theta - \pi \leq 4\pi \Leftrightarrow 0 \leq 2\theta \leq 5\pi$$

$$\Leftrightarrow 0 \leq \theta \leq \frac{5\pi}{2}$$

Solving for $\cos x = 0$: $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

Solving for $\sin x = \frac{3}{4}$: $x = \sin^{-1}\left(\frac{3}{4}\right), \pi - \sin^{-1}\left(\frac{3}{4}\right)$

We solved two separate equations, giving the solution, $x = 0, \sin^{-1}\left(\frac{3}{4}\right), \frac{\pi}{2}, \pi - \sin^{-1}\left(\frac{3}{4}\right), \frac{3\pi}{2}$

c This time we make use of the cosine double-angle formula, $\cos 2x = 1 - 2\sin^2 x$.

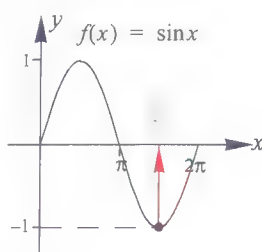
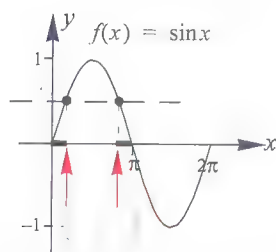
$$\therefore \sin x = \cos 2x$$

$$\Leftrightarrow \sin x = 1 - 2\sin^2 x$$

$$\Leftrightarrow 2\sin^2 x + \sin x - 1 = 0$$

$$\Leftrightarrow (2\sin x - 1)(\sin x + 1) = 0$$

$$\Leftrightarrow \sin x = \frac{1}{2} \text{ or } \sin x = -1$$



Again, we have two equations to solve.

Solving for $\sin x = \frac{1}{2}$:

Solving for $\sin x = -1$:

Therefore, solution set is $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$.

As the next example shows, the working required to solve trigonometric equations can be significantly reduced, especially if you know the exact values for the basic trigonometric angles as well as the symmetry properties (without making use of a graph). However, we encourage you to use a visual aid when solving such equations.

Example C.7.6

Solve the equation $4\sin x \cos x = \sqrt{3}, -2\pi \leq x \leq 2\pi$

$$4\sin x \cos x = \sqrt{3}, -2\pi \leq x \leq 2\pi$$

$$2\sin 2x = \sqrt{3} \text{ using } \sin 2\theta = 2\sin \theta \cos \theta$$

$$\sin 2x = \frac{\sqrt{3}}{2}$$

$$2x = \dots, -\frac{11\pi}{3}, -\frac{10\pi}{3}, -\frac{5\pi}{3}, -\frac{4\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}, \dots$$

$$x = -\frac{11\pi}{6}, -\frac{5\pi}{3}, -\frac{5\pi}{6}, -\frac{2\pi}{3}, \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$$

Exercise C.7.1

1. If $0 \leq x \leq 2\pi$, find:

a $\sin x = \frac{1}{\sqrt{2}}$

b $\sin x = -\frac{1}{2}$

c $\sin x = \frac{\sqrt{3}}{2}$

d $\sin 3x = \frac{1}{2}$

e $\sin\left(\frac{x}{2}\right) = \frac{1}{2}$

f $\sin(\pi x) = -\frac{\sqrt{2}}{2}$

2. If $0 \leq x \leq 2\pi$, find:

a $\cos x = \frac{1}{\sqrt{2}}$

b $\cos x = -\frac{1}{2}$

c $\cos x = \frac{\sqrt{3}}{2}$

d $\cos\left(\frac{x}{3}\right) = \frac{1}{2}$

e $\cos(2x) = \frac{1}{2}$

f $\cos\left(\frac{\pi}{2}x\right) = -\frac{\sqrt{2}}{2}$

3. If $0 \leq x \leq 2\pi$, find:

a $\tan x = \frac{1}{\sqrt{3}}$

b $\tan x = -1$

c $\tan x = \sqrt{3}$

d $\tan\left(\frac{x}{4}\right) = 2$

e $\tan(2x) = -\sqrt{3}$

f $\tan\left(\frac{\pi}{4}x\right) = -1$

4. If $0 \leq x \leq 2\pi$ or $0 \leq x \leq 360$, find:

a $\sin(x^\circ + 60^\circ) = \frac{1}{2}$

b $\cos(x^\circ - 30^\circ) = -\frac{\sqrt{3}}{2}$

c $\tan(x^\circ + 45^\circ) = -1$

d $\sin(x^\circ - 20^\circ) = \frac{1}{\sqrt{2}}$

e $\cos\left(2x - \frac{\pi}{2}\right) = \frac{1}{2}$

f $\tan\left(\frac{\pi}{4}x\right) = 1$

g $\sec(2x + \pi) = 2$

h $\cot\left(2x + \frac{\pi}{2}\right) = 1$

5. If $0 \leq x \leq 2\pi$ or $0 \leq x \leq 360$, find:

a $\cos x^\circ = \frac{1}{2}$

b $2\sin x + \sqrt{3} = 0$

c $\sqrt{3}\tan x = 1$

d $5\sin x^\circ = 2$

e $4\sin^2 x - 3 = 0$

f $\frac{1}{\sqrt{3}} \tan x + 1 = 0$

g $2\sin\left(x + \frac{\pi}{3}\right) = -1$

h $5\cos(x+2) - 3 = 0$

i $\tan\left(x - \frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$

j $2\cos 2x + 1 = 0$

k $\tan 2x - \sqrt{3} = 0$

l $2\sin x^\circ = 5\cos x^\circ$

m $2\operatorname{cosec}\left(\frac{x}{2}\right) = 4$

n $\frac{1}{2} \cot(2x) = 0$

o $\sec\left(\frac{x}{3}\right) = -\sqrt{2}$

6. Solve the following equations for the intervals indicated, giving exact answers:

a $\sin \theta \cos \theta = \frac{1}{2}, -\pi \leq \theta \leq \pi$

b $\cos^2 \theta - \sin^2 \theta = -\frac{1}{2}, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

c $\tan A = \frac{1 - \tan^2 A}{2}, -\pi \leq A \leq \pi$

d $\frac{\sin \theta}{1 + \cos \theta} = -1, -\pi \leq \theta \leq \pi$

e $\cos^2 x = 2\cos x, -\pi \leq x \leq \pi$

f $\sec 2x = \sqrt{2}, 0 \leq x \leq 2\pi$

7. Solve each of the following equations in the specified domain.

a $2\sin^2 x + \sin x - 1 = 0, 0 \leq x \leq 4\pi$

b $\cos^2 x + 2\cos x - 3 = 0, -2\pi \leq x \leq 2\pi$

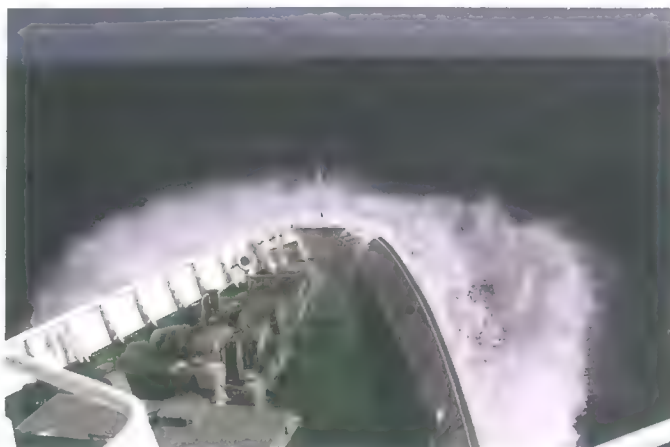
c $2\sin^2 x - \sin x = 3, -\pi \leq x \leq 5\pi$

d $2\sin^2 x - 5\sin x + 2 = 0, 0 \leq x \leq 6\pi$

e $6\cos^2 x + 5\cos x + 1 = 0, -2\pi \leq x \leq 4\pi$

Applications

Functions of the type considered in the previous section are useful for modelling periodic phenomena.



The pitching and rolling of a ship in a heavy sea.



The flexing of large structures such as suspension bridges.



The vibration of aircraft wings.

Extra questions



These sorts of applications usually start with data that has been measured in an experiment. The next task is to find a function that 'models' the data in the sense that it produces function values that are similar to the experimental data. Once this has been done, the function can be used to predict values that are missing from the measured data (interpolation) or values that lie outside the experimental data set (extrapolation).

Example C.7.7

The table shows the depth of water at the end of a pier at various times (measured, in hours after midnight on the first day of the month.)

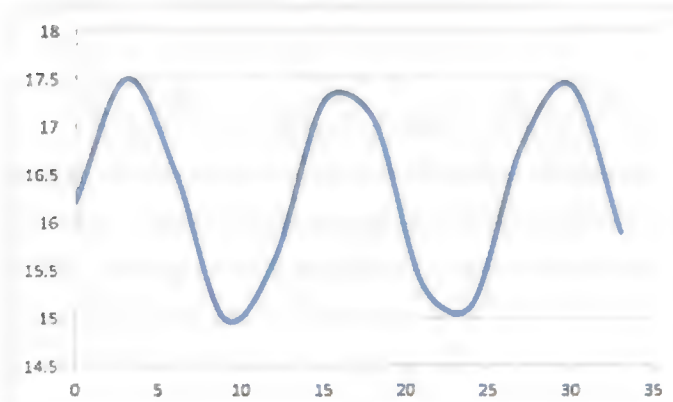
t (hr)	0	3	6	9	12	15	18
d m	16.20	17.49	16.51	14.98	15.60	17.27	17.06

t (hr)	21	24	27	30	33
d m	15.34	15.13	16.80	17.42	15.89

Use your model to predict the time of the next high tide.

Plot the data as a graph. Use your results to find a rule that models the depth data.

We start by entering the data as lists and then plotting them using graph paper or a graph plotter. This graph was produced using a spreadsheet with 'scatter/smooth line' selected.

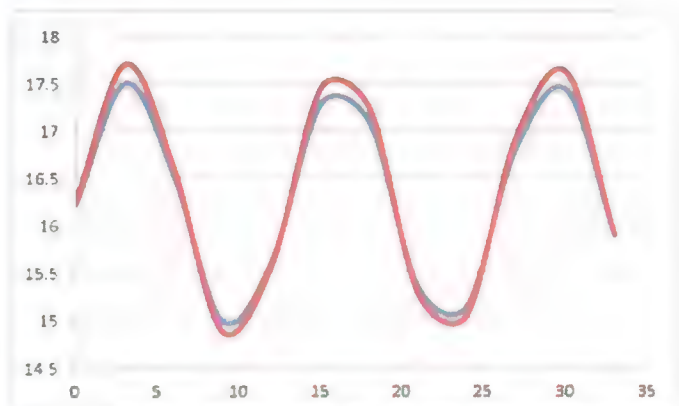


This does suggest that the depth is varying periodically. It appears that the period is approximately 13 hours. This is found by looking at the time between successive high tides. This is not as easy as it sounds as the measurements do not appear to have been made exactly at the high tides. This means that an estimate will need to be made based upon the observation that successive high tides appear to have happened after 3, 16 and 32 hours. Next, we look at the amplitude and vertical translation. Again, because we do not have exact readings at high and low tides, these will need to be estimated. The lowest tide recorded is 14.98 and the highest is 17.49.

A first estimate of the vertical translation is $\frac{17.49 + 14.98}{2} = 16.235$ and the amplitude is

$17.7 - 16.235 = 1.465$. Since the graph starts near the mean depth and moves up it seems likely that the first model to try might be:

$$y = 1.465 \times \sin\left(\frac{2\pi t}{13}\right) + 16.235$$



A possible modelling function is shown in red.

Notice that the dilation factor (along the x -axis) is found by using the result that if

$$\tau = 13 \Rightarrow \frac{2\pi}{n} = 13 \therefore n = \frac{2\pi}{13}$$

The model should now be 'evaluated' which means testing how well it fits the data. This can be done by making tables of values of the data and the values predicted by the model and working to make the differences between these as small as possible. This can be done using a scientific or graphics calculator.

The model shown is quite good as errors are small with some being positive and some being negative. The function used is $y = 1.465 \times \sin\left(\frac{2\pi t}{13}\right) + 16.235$ and this can now be used to predict the depth for times that measurements were not made. Also, the graph of the modelling function can be added to the graph of the data (as shown).

The modelling function can also be used to predict depths into the future (extrapolation).

The next high tide, for example can be expected to be 13 hours after the previous high tide at about 29.3 hours. This is after 42.3 hours.

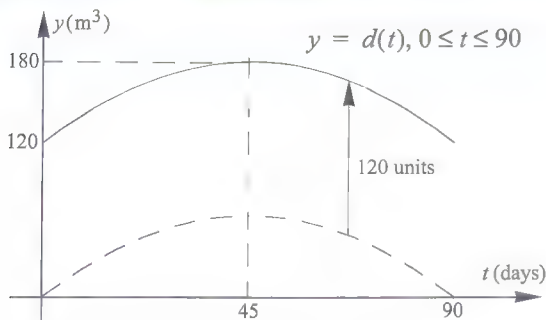
Example C.7.8

A reservoir supplies water to an outer suburb based on the water demand, $D(t) = 120 + 60 \sin\left(\frac{\pi}{90}t\right)$, $0 \leq t \leq 90$, where t measures the number of days from the start of summer (which lasts for 90 days).

Sketch the graph of $D(t)$.

What are the maximum and minimum demands made by the community over this period?

a



The features of this function are:

$$\text{Period} = \frac{2\pi}{\left(\frac{\pi}{90}\right)} = 180 \text{ days}$$

$$\text{Amplitude} = 60$$

$$\text{Translation} = 120 \text{ units up.}$$

We 'pencil in' the graph of $y = 60 \sin\left(\frac{\pi}{90}t\right)$ and then

move it up 120 units:

b The minimum is 120m^3 and the maximum is 180m^3 .

a The amplitude is 20 mmHg and the period is given by:

$$\frac{2\pi}{\left(\frac{5\pi}{3}\right)} = \frac{6}{5} = 1.2 \text{ seconds.}$$

b The maximum is given by:

$$(100 + \text{amplitude}) = 100 + 20 = 120.$$

c One full cycle is 1.2 seconds long:

d



Note that the graph has been drawn as opposed to sketched. That is, it has been accurately plotted, meaning that the scales and the curve are 'accurate'. Because of this we can read directly from the graph.

In this case, $P = 110$ when $t = 0.4$ and 0.8 .

Even though we have drawn the graph, we will now solve the relevant equation:

$$P(t) = 110 \Leftrightarrow 110 = -20 \cos\left(\frac{5\pi}{3}t\right) + 100$$

$$\Leftrightarrow 10 = 20 \cos\left(\frac{5\pi}{3}t\right)$$

$$\Leftrightarrow \cos\left(\frac{5\pi}{3}t\right) = \frac{1}{2}$$

$$\therefore \frac{5\pi}{3}t = \pi - \cos^{-1}\left(\frac{1}{2}\right), \pi + \cos^{-1}\left(\frac{1}{2}\right)$$

$$\Leftrightarrow \frac{5\pi}{3}t = \frac{2\pi}{3}, \frac{4\pi}{3}$$

$$\Leftrightarrow t = \frac{2}{5}, \frac{4}{5}$$

Example C.7.9

When a person is at rest, the blood pressure, P millimetres of mercury at any time t seconds, can be approximately modelled by the equation:

$$P(t) = -20 \cos\left(\frac{5\pi}{3}t\right) + 100, t \geq 0$$

a Determine the amplitude and period of P .

b What is the maximum blood pressure reading that can be recorded for this person?

c Sketch the graph of $P(t)$, showing one full cycle.

d Find the first two times when the pressure reaches a reading of 110 mmHg.

Exercise C.7.2

1. The table shows the temperature in an office block over a 36-hour period.

t (hr)	0	3	6	9	12	15	18
$T^{\circ}\text{C}$	18.3	15.0	14.1	16.0	19.7	23.0	23.9

t (hr)	21	24	27	30	33	36
$T^{\circ}\text{C}$	22.0	18.3	15.0	14.1	16.0	19.7

- a Estimate the amplitude, period, horizontal and vertical translations.
- b Find a rule that models the data.
- c Use your rule to predict the temperature after 40 hours.

2. The table shows the light level L during an experiment on dye fading.

t (hr)	0	1	2	3	4	5
L	6.6	4.0	7.0	10.0	7.5	4.1

t (hr)	6	7	8	9	10
L	6.1	9.8	8.3	4.4	5.3

- a Estimate the amplitude, period, horizontal and vertical translations.
- b Find a rule that models the data.
3. The table shows the value in \$ of an industrial share over a 20-month period.

Month	0	2	4	6	8	10
Value	7.0	11.5	10.8	5.6	2.1	4.3

Month	12	14	16	18	20
Value	9.7	11.9	8.4	3.2	2.5

- a Estimate the amplitude, period, horizontal and vertical translations.
- b Find a rule that models the data.
4. The table shows the population (in thousands) of a species of fish in a lake over a 22-year period.

Year	0	2	4	6	8	10
Pop	11.2	12.1	13.0	12.7	11.6	11.0

Year	12	14	16	18	20	22
Pop	11.6	12.7	13.0	12.1	11.2	11.2

- a Estimate the amplitude, period, horizontal and vertical translations.

- b Find a rule that models the data.

5. The table shows the average weekly sales (in thousands of \$) of a small company over a 15-year period.

Time	0	1.5	3	4.5	6	7.5
Sales	3.5	4.4	7.7	8.4	5.3	3.3

Time	9	10.5	12	13.5	15
Sales	5.5	8.5	7.6	4.3	3.6

- a Estimate the amplitude, period, horizontal and vertical translations.
- b Find a rule that models the data.

6. The table shows the average annual rice production, P , (in thousands of tonnes) of a province over a 10-year period.

t (yr)	0	1	2	3	4	5
P	11.0	11.6	10.7	10.5	11.5	11.3

t (yr)	6	7	8	9	10
P	10.4	11.0	11.6	10.7	10.5

- a Estimate the amplitude, period, horizontal and vertical translations.
- b Find a rule that models the data.

7. The table shows the depth of water (D metres) over a 5-second period as waves pass the end of a pier.

t (sec)	0	0.5	1	1.5	2
D	11.3	10.8	10.3	10.2	10.4

t (sec)	2.5	3	3.5	4	4.5	5
D	10.9	11.4	11.7	11.8	11.5	11.0

- a Estimate the amplitude, period, horizontal and vertical translations.
- b Find a rule that models the data.
8. The population (in thousands) of a species of butterfly in a nature sanctuary is modelled by the function:

$$P = 3 + 2 \sin\left(\frac{3\pi t}{8}\right), 0 \leq t \leq 12$$

where t is the time in weeks after scientists first started making population estimates.

- a What is the initial population?
- b What are the largest and smallest populations?
- c When does the population first reach 4 thousand butterflies?

9. A water wave passes a fixed point. As the wave passes, the depth of the water (D metres) at time t seconds is modelled by the function:

$$D = 7 + \frac{1}{2} \cos\left(\frac{2\pi t}{5}\right), t > 0$$

- a What are the greatest and smallest depths?
 - b Find the first two times at which the depth is 6.8 metres.
10. The weekly sales (S) (in hundreds of cans) of a soft drink outlet is modelled by the function:

$$S = 13 + 5.5 \cos\left(\frac{\pi t}{6} - 3\right), t > 0$$

t is the time in months with $t = 0$ corresponding to 1st January 1990.

- a Find the minimum and maximum sales during 1990.
 - b Find the value of t for which the sales first exceed 1500 ($S = 15$).
 - c During which months do the weekly sales exceed 1500 cans?
11. The rabbit population, $R(t)$ thousands, in a northern region of South Australia is modelled by the equation:

$$R(t) = 12 + 3 \cos\left(\frac{\pi t}{6}\right), 0 \leq t \leq 24,$$

where t is measured in months after the first of January.

- a What is the largest rabbit population predicted by this model?
- b How long is it between the times when the population reaches consecutive peaks?
- c Sketch the graph of $R(t)$ for $0 \leq t \leq 24$.
- d Find the longest time span for which $R(t) \geq 13.5$.
- e Give a possible explanation for the behaviour of this model.

12. Samantha is sitting in a rocking chair. The distance $d(t)$, in centimetres, between the wall and the rear of the chair varies sinusoidally with time t , in seconds. At time $t = 1$, the chair is closest to the wall and $d(1) = 18$. At $t = 2$, the chair is farthest from the wall and $d(2) = 34$.

- a What is the period of the function?
- b How far is the chair from the wall when no one is rocking in it?
- c What is the domain of the function, if Samantha rocks the chair back and forth 20 times?
- d What is the range of the function?
- e What is the amplitude of the function?
- f What is the equation of the sinusoidal function?
- g What is the distance between the wall and the chair at $t = 8$ s?

13. A weight attached to the end of a long spring is bouncing up and down. As it bounces, its distance from the floor varies sinusoidally with time. You start a stopwatch. When the stopwatch reads 0.4s, the weight first reaches a high point 72cm above the floor. The next low point, 32cm above the floor, occurs at 1.6s.

- a Find a particular equation for distance from the floor as a function of time.
- b What is the distance from the floor when the stopwatch reads 17.2s?
- c What was the distance from the floor when you started the stopwatch?
- d What is the first positive value of time when the weight is 59cm above the floor?

Extra
questions



Answers



SECTION FOUR

STATISTICS AND PROBABILITY





Regression

In Chapter D.4 of the Common Core text we introduced the idea of bivariate data and the idea that two (sometimes seemingly unconnected) data sets may be connected. Our picture is of a street market in Morocco where the price of goods is negatively correlated with their supply. Here is a further example to remind you of the main ideas.

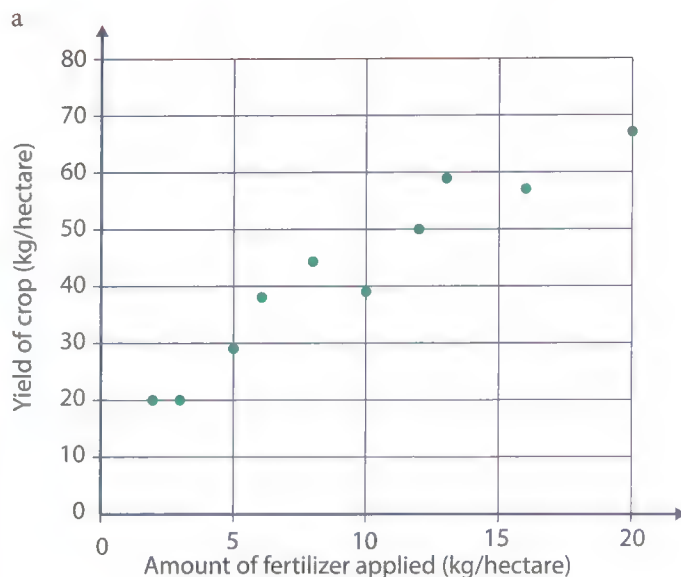
Example D.4.1

An Agricultural Supply Company has developed a new fertiliser. Field tests were undertaken in which the observed yield, y kilograms per unit area of a crop, was measured for various amounts of fertilizer applied, x kilograms per hectare. The data below give observed yields for various values of x :

x	2	12	3	16	5	6	20	8	13	10
y	20	50	20	57	29	38	67	44	59	39

- Draw a scatter diagram for the given set of data.
- Calculate the correlation coefficient, r , for the given data set. Is this an appropriate statistic to calculate for this data set?
- Use the method of least squares to determine the line of best fit.

A scatter diagram (scattergraph) shows these data on what is, effectively, a Cartesian Graph. Note that it is not legitimate to join the points.



At this stage, we have only described correlations in fairly imprecise terms. Nevertheless, these can be immediately useful. We can, for example, make these direct inferences.

- because the points lie on a 'bottom left to top right' diagonal, we can say that there is a 'positive correlation'. Larger amounts of fertilizer produce heavier crop yields.
- on this basis, it is legitimate for the company to claim that the fertilizer 'works'.
- the points are fairly close to a straight line and we would, therefore, describe the correlation as strong, further backing the claim that the fertilizer 'works'.

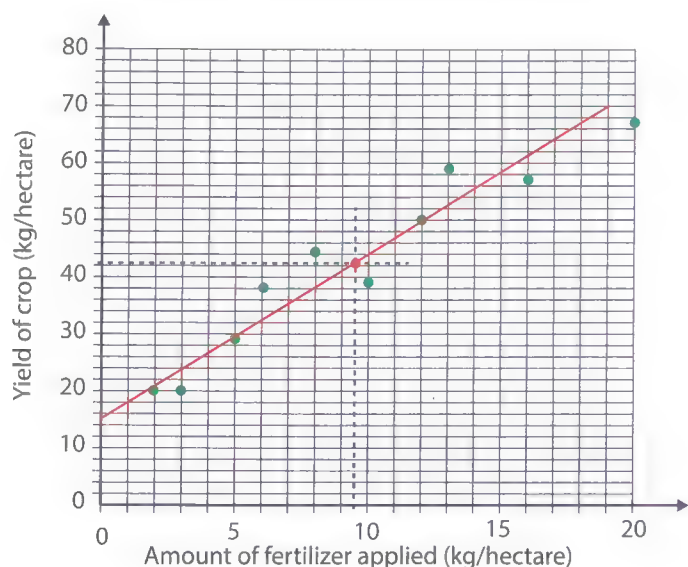
- b Finding the Pearson Correlation Coefficient (r) using technology was described in the Core Text D.4. The value of r^2 in this case is: 0.92228 so $r \approx 0.96$, a strong positive correlation. This is relevant here as it gives objective backing to the claims made in the dot points.
- c So far, we have mostly looked at straight lines to fit such data sets that have been drawn 'by eye'. This has been aided by calculating a mean point and using it as a fixed point about which we can pivot possible lines of best fit.

There are ten data pairs.

The x values sum to 95 and have a mean of 9.5

The y values sum to 423 and have a mean of 42.3

The mean point (red dot) has been used as a pivot and the line of best fit (also red) has been chosen so that there are three points on the line, three above it and four beneath. It is also a reasonable fit 'by eye'.



Equation of Line of Best Fit

In these applications we are only looking for linear fitting lines. These will always, therefore, have equations of the form $y = mx + c$.

The gradient is found by drawing a gradient triangle. If we take the full extent of the red line in the above diagram it has a rise of 15 to 70 (=55) and a run of 0 to 19 (=19). This gives a gradient (rise÷run) of $55 \div 19 \approx 2.9$.

The intercept can be read directly from the graph (19). The line of best fit has equation $y = 2.9x + 15$.

In the Core Text D.4, we touched on the use of technology to find r and the equation of the line of best fit.

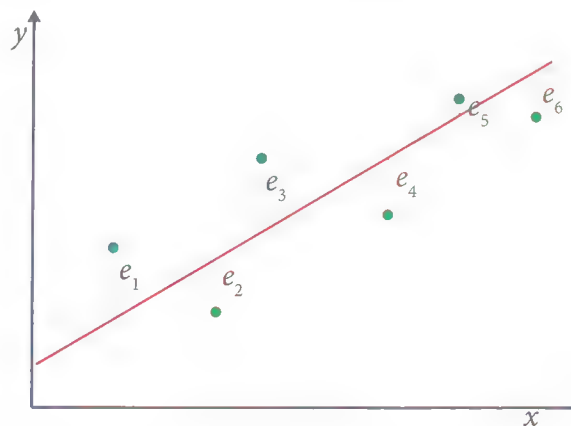
We will now look in a bit more detail at how such calculations are done and what inferences we can draw from them.

When we find one of these lines of best fit, by whatever method, there will be errors. The line will not, except in rare circumstances, go through all the data points.

Method of Least Squares

We have considered a number of alternatives to determine the line of best fit. So far, they have all had an element of 'guess work'. Obtaining a line of best fit that adheres to sound statistical procedures is done by using a simple linear regression method.

There are a number of different ways this can be achieved. However, we will consider the method known as 'the method of least squares'. The least squares regression line is the line which makes the sum of the squares of the vertical deviations of the data from the line as small as possible:



That is, having drawn the scatter diagram, we want to determine the equation $y = mx + c$ for which the sum of the squares of the errors, e_i , between observed values and predicted values (based on the straight line) is minimal, i.e. we want to minimize the expression:

$$\Delta = \sum_{i=1}^n e_i^2 = \sum_{i=1}^n (y_i - mx_i - c)^2$$

Although the mathematics is beyond the scope of this course, we can obtain a rather neat expression that will enable us to determine the value of m and c that give the line of best fit.

We quote this result now.

$$c = \bar{y} - m\bar{x} \text{ and } m = \frac{\sum xy - \frac{1}{n}(\sum x)(\sum y)}{\sum x^2 - \frac{1}{n}(\sum x)^2}$$

Again, it can be seen from these formulae that manual calculations are very tedious, and so it is highly recommended that graphics calculators are used (or any calculator that handles bivariate data).

Once we obtain the line of regression of y on x , it is important that we not only use it to predict values but also to interpret the intercept and the slope of the line in terms of the problem being discussed.

When dealing with regression lines, predictions take on the form of interpolation and extrapolation. **Interpolation** is the process of using a regression line to make predictions within the range of data that was used to derive the equation, whereas **extrapolation** is the process of using a regression line to make predictions outside the range of data that was used to derive the equation.

Example D.4.1

Using the data from Example D.4.1:

- a Use a calculator to find the correlation coefficient, r , and the equation of the line of best fit.
- b Use your equation to predict the crop yield if the fertilizer is applied at these rates:
 - i 11 kilograms per hectare.
 - ii 23 kilograms per hectare.
 - iii 38 kilograms per hectare.
- a The exact method will depend on the model of calculator that you are using. Almost always, the data is entered as two lists. Review the video attached to the D.4 chapter in the Core text:



You should get $r^2 \approx 0.922$ and $r \approx 0.960$. This means that about 92% of the variation is due to the actual connection between the yield and the application of fertilizer. The regression line $y = 2.68x + 16.9$.

- b Now that we have established a strong correlation, it is legitimate to use the regression line to make predictions.

- i If $x = 11$ kg per hectare.

$$y = 2.68 \times 11 + 16.9 \approx 46 \text{ kg per hectare.}$$

This is within the data set (interpolation) and is a legitimate prediction. Note that, in common with many calculations of this sort, we have rounded the answer to a level (2 s.f.) that is similar to the accuracy level of the data.

- ii If $x = 23$ kg per hectare.

$$y = 2.68 \times 23 + 16.9 \approx 79 \text{ kg per hectare.}$$

This is just outside the data set and represents a reasonable extrapolation.

- iii If $x = 38$ kg per hectare.

$$y = 2.68 \times 38 + 16.9 \approx 119 \text{ kg per hectare.}$$

This is a long way outside the data set and represents an unreasonable extrapolation. How do we distinguish between reasonable and unreasonable extrapolations? At this level, common sense will do. There are good reasons to suppose that the increased yields will not go on for ever as more fertilizer is added to the soil. There is probably a point at which it will disrupt soil chemistry and produce reductions in yield.

Example D.4.2

In a test of a new drug for controlling high blood pressure, varying doses were given to a group of patients all with systolic pressures in the range 150–160 mmHg. The results were:

d	5	10	15	20	25	30
S	165	142	158	125	138	120

S is the mean systolic pressure of the group when a dose of d mg was administered.

Perform a least squares regression analysis on the data and comment on the results. Can the analysis be used to predict the effect of doses of 17 mg? Or 50 mg?

- a A calculator analysis gives:

	Deg	Norm1	d/c	Real
LinearReg(ax+b)				
a = -1.5428571				
b = 168.333333				
r = -0.8141053				
r ² = 0.66276744				
MSe = 132.47619				
y = ax + b				
				COPY

The correlation is negative (ie. the drug does appear to reduce blood pressure). However, $r^2 \approx 0.66$ indicates that about 66% of the effect is due to drug dosage. This should indicate some caution when using the study to make predictions, particularly outside the data set.

The linear regression line is $S = -1.54d + 168$.

The prediction for $d = 17$ mg is $S \approx 142$ mmHg. This has some legitimacy, but the level of certainty is not high.

The prediction for $d = 50$ mg is $S \approx 91$ mmHg. This has no legitimacy as it is a large extrapolation using an already fairly weak case.

The safest conclusion here is that the study probably needs to be redesigned and repeated. Was the sample large enough etc.?

2. The content of sand in soil at different depths from a particular area was recorded and tabulated.

x	0	12	24	36	48	60	72	84	96
y	80	63	64	62	57	59	41	47	38

where x (depth in cm) and y (% sand).

- a Draw a scatter diagram for the given set of data.
- b Calculate the correlation coefficient for the given data set. Is this an appropriate statistic to calculate for this data set?
- c i Use the method of least squares to determine the line of best fit.
- ii Graph the regression line on the scatter diagram in part a.
- d Using your line of best fit, determine the percentage of sand in the soil at a depth:
- i of 40 cm. Would this be considered as extrapolation or interpolation?
- ii of 120 cm. Would this be considered as extrapolation or interpolation?

3. The school nurse at Queens Hill High School is conducting a study into the relationship between the height and weight of school students. Everyday she weighs and measures students that attend her room. One particular day, 8 students need her attention. She carries out her study with the following results.

h	130	120	155	160	140	140	165	135	150
w	45	40	60	61	60	54	70	50	58

where h = Height (cm) and w = Weight (kg)

- a Plot the scatter diagram for this data set.
- b Use the method of least squares to determine the line of best fit.
- c Calculate the product moment correlation coefficient for this data. What can you conclude from this value?

Exercise D.4.1

1. The relationship between the temperature, T °C of water and the mass, M kg, of a chemical substance dissolving in the water has been tabulated:

T °C	10	20	30	40	50	60	70	80	90
M kg	50	53	56	61	66	69	70	71	72

- a Draw a scatter diagram for the given data set.
- b Find the coefficient of correlation.
- c Find the least squares regression line which best fits the data.

4. Which bivariate sets X and Y would be expected to display a Pearson's correlation coefficient closest to -1 ?
- A. X : the number of words in a book Y : the number of pages in a book.
- B. X : the time taken to cycle 10 km Y : the average speed of the cyclist.
- C. X : the face up value of a die Y : the face up value of a die Y in a set of trials where the two dice X and Y are rolled 100 times.
- D. X : students' test marks in Chemistry Y : students' test marks in Mathematics.
- E. X : Maximum daily temperature in Wales Y : Minimum daily temperature in Wales.

5. The data in the following table shows the percentage marks in Mathematics (M), English (E) and Physics (P) for a group of students in 2004.

Student	M	E	P
1	100	88	88
2	72	75	65
3	86	88	88
4	88	72	79
5	98	86	94
6	91	68	78
7	72	63	67
8	98	91	100
9	100	91	100
10	75	64	62
11	75	63	65
12	84	82	73

- a Calculate, correct to two decimal places, the product moment correlation coefficient between Maths marks and English marks.
- b i Find the equation of the two means regression line of P on M , giving all constants correct to two decimal places.
- ii Predict the Physics mark for a student who scored 80 in Mathematics.
- c i Find the least squares regression equation of M on E .

- ii Predict the Maths mark for a student who scored 95 in English.
- iii In part ii, did you interpolate or extrapolate? How confident are you in your prediction?
- d If you wished to predict the English marks for a student who had scored 90 in Mathematics, how would you go about doing this.

6. For the data below describe the type of relationship that exists between the variables X and Y , including:

x	1	9	7.5	10	2.5	5.0	11	5.5	14	6
y	10	47	45	60	20	35	65	23	68	35

- a the direction
- b the form
- c the strength of relationship.

7. The table shows the income (in thousands of dollars) and the annual expenditure, in hundreds of dollars, for ten single male workers aged 30–40 years.

Income = I and Expenditure = E .

I	42	34	36	38	40	39	36	38	39	38
E	55	39	47	49	55	53	42	44	50	51

- a i Plot the data on a scatter diagram.
- ii Find the correlation coefficient.
- b Calculate the proportion of the variance of Expenditure which can be explained by the variance of the Income.
- c Find the least squares equation of the regression line.
- d On the scatter diagram from part a, sketch the regression line.
- e Estimate the expenditure by a single male working person aged 30–40 years if their annual income is \$37 000.

8. A farmer is trying to establish a relationship between the final weight, x lbs, and the carcass weight, y lbs, for a particular type of bull. From a pen of 10 such bulls, she obtained the following weights.

x	y
1030	614
1000	577
1060	654
980	594
995	593
1025	589
1055	629
1035	650
1380	834
1085	691

- a i Plot the data on a scatter diagram.
- ii Find the correlation coefficient.
- b Find the least squares equation of the regression line.
- c On the scatter diagram from (a), sketch the regression line.
- d Estimate the carcass weight of a bull if its final weight is known to be 1040 lbs.



One of the most important distinctions to be made when looking at correlation is to be clear that establishing a correlation does not imply **causation**.

The fact that one quantity is mathematically linked to another does not imply that changing one actually causes the other to change.

If you run an internet search on something like 'spurious correlation' you will find websites that list some very strange correlations.

There is, apparently, a strong positive correlation between the annual rate of drownings in US swimming pools and the number of films featuring the actor Nicholas Cage. Nobody, one hopes, would conclude from this that Nicholas Cage films actually cause pool drownings!

Some sites that you may find in your searches:

<https://www.neatorama.com/2014/05/10/Spurious-Correlations/>



<http://www.tylervigen.com/spurious-correlations>



Whilst it is entertaining to look at these spurious cases, there is a more subtle form of correlation that you might like to consider as a subject for your investigation.

This relates to situations in which the 'causation factor' is hidden.

For example, there is some positive correlation between a person's expenditure on cigarettes and their expenditure on gambling. It is unlikely that one of these factors directly causes the other. However, there is some interest in the idea that there might be a hidden third factor that causes both a tendency to smoke and a tendency to gamble. If there is an 'attraction to addictive behaviour' factor that causes both, it could affect the way society views such problems.

Answers





Correlation techniques can be applied to a range of problems.

You are encouraged to look for an application that interests you particularly.

As an example, it is sometimes said that we inherit our weather from the places to the west of us. This is because weather patterns tend to move with the prevailing winds. In many parts of the World, these blow from west to east.

The table shows the top temperatures ($^{\circ}\text{C}$) in Chicago and New York over a block of days in February 2019.

Date	Chicago	New York
7	8	7
8	-9	12
9	-4	0
10	-2	2
11	0	3
12	1	2
13	-1	4
14	7	7
15	-2	14
16	-1	7
17	-1	2
18	-1	6
19	0	2
20	2	1
21	1	9

Chicago is about 1 250 km to the West of New York.

The cities are sufficiently separated for it to be unlikely that the weather conditions in both places will be the similar on any one day.

But does New York 'inherit' Chicago's temperatures a day later? Two days later? Not at all?

If you suspect that there is a 'one day later' relationship, you will need to look at data pairings that are displaced by one day (as in the green cells).

If you suspect a 'two days later' relationship, you will need to displace by two days.

Remember also the distinction between equality and correlation!

There are many other measures of weather that might be chosen, barometric pressure, humidity, rainfall etc.

D.5 Conditional Probability

SL 4:117

In Chapter D5 of the Common Core text, we looked at the general ideas of probability. Mathematical probability was devised to formalise the calculation of odds in casino games and to assist in the fixing of insurance premiums.

The first of these is known as **theoretical probability** and usually depends on counting events.

The second is loosely termed **experimental probability** and depends on experience. The people who quantify this 'experience' are known as actuaries. Essentially, if they are asked to quote a motor insurance premium, they ask the applicant a number of key questions that have been found to indicate the chances that a driver will have an accident. These include questions such as age, gender etc., not because they are obviously connected to the chance that a driver will have an accident. The reason is that experience shows that they are.

Mathematics courses tend to concentrate on theoretical probability and this is no exception. We begin by summarising the basic ideas. Consider this set of ten playing cards:



The set of all the possible outcomes if a single card is drawn is known as the **sample space** of the problem. When solving probability problems, identifying the sample space is almost always the first step. In the case of the cards in the picture:

Sample Space = {Q♠, A♥, 2♥, K♦, 3♥, 10♣, 9♥, A♣, 6♥, K♠}.

The number of elements in this sample space is 10.

If a card is selected at random, what is the probability that it is red. This is answered by counting the number of red cards in the sample space.

$$\begin{aligned} P(\text{red}) &= \frac{\text{number of red cards}}{\text{number of cards}} \\ &= \frac{6}{10} \\ &= 0.6 \end{aligned}$$

Similarly, the probability of drawing an ace:

$$\begin{aligned} P(\text{ace}) &= \frac{\text{number of aces}}{\text{number of cards}} \\ &= \frac{2}{10} \\ &= 0.2 \end{aligned}$$

There are plenty of questions in which the counting of the outcomes is not as straightforward as it is in this case.

It is probably a good moment to review the use of:

- Venn Diagrams
- Tree Diagrams
- Lattice Diagrams.

We have also had a brief look at the idea of Conditional Probability.

An example of a conditional question relating to our set of cards is:

'Find the probability that a card chosen at random is an Ace, given that it is red'.

There are two approaches. So far, we have concentrated on answering these questions by looking at a restricted sample space. In this case, we are told that the card chosen is red so:



Sample Space = $\{A♥, 2♥, K♦, 3♥, 9♥, 6♥\}$.

$$P(\text{ace}) = \frac{\text{number of red aces}}{\text{number of red cards}} = \frac{1}{6}$$

Conditional Probability Formula

If A and B are two events, then **the conditional probability of event A given event B** is found using:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, P(B) \neq 0$$

Note:

1. If A and B are mutually exclusive then: $P(A|B) = 0$
2. From the above rule, we also have the general Multiplication rule:

$$P(A \cap B) = P(A|B) \times P(B).$$

It should also be noted that usually $P(A|B) \neq P(B|A)$.

Recall that events are exclusive if they cannot occur together. A coin cannot simultaneously fall heads AND tails. However A = 'it is raining' and B = 'the sun is shining'. A and B are not exclusive as there are occasional 'sun showers'.

The use of the formula to solve our problem: 'Find the probability that a card chosen at random is an Ace, given that it is red' does not follow the same path as the restricted domain method.

If using the formula, it is important to define the two events.

A = the card chosen is an ace.

B = the card chosen is red.

$A \cap B$ = the card chosen is a red ace.

$$P(A \cap B) = \frac{\text{number of red aces}}{\text{number of cards}} = \frac{1}{10}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, P(B) \neq 0$$

$$P(B) = \frac{\text{number of red cards}}{\text{number of cards}} = \frac{6}{10}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{10}}{\frac{6}{10}} = \frac{1}{6}$$

Example D.5.1

An urn contains 10 blue balls, 6 black balls and 4 red balls.

A ball is drawn, its colour noted. Find the probabilities that the balls are:

- a both blue.
- b red then blue.
- c blue then black.

The sample space at the start is: $\{10\bullet, 6\bullet, 4\bullet\}$.

- a The probability of drawing a blue at the first draw

$$= \frac{10}{20}$$

The sample space is now $\{9\bullet, 6\bullet, 4\bullet\}$.

The probability of drawing a second blue ball $= \frac{9}{19}$.

Since the events are independent (both draws are at random):

$$P(2 \text{ blue balls}) = \frac{10}{20} \times \frac{9}{19} = \frac{9}{38}$$

b Red then blue.

$P(\text{red}) = \frac{4}{20}$. The sample space is now $\{10\bullet, 6\bullet, 3\bullet\}$.

$$P(\text{blue}) = \frac{10}{19}$$

$$P(\text{red then blue}) = \frac{4}{20} \times \frac{10}{19} = \frac{4}{38}$$

c Blue then black

$P(\text{blue}) = \frac{10}{20}$ The sample space is now $\{9\bullet, 6\bullet, 4\bullet\}$.

$$P(\text{black}) = \frac{6}{19}$$

$$P(\text{blue then black}) = \frac{10}{20} \times \frac{6}{19} = \frac{3}{19}$$

Example D.5.2

Two fair dice are rolled. Find the probability that the numbers on the faces are the same given that the dice show a sum of 4.

You might like to use a lattice diagram and think of the dice as having different colours.

		Red Die A					
		1	2	3	4	5	6
Blue Die B	1	●	●	●	●	●	●
	2	●	●	●	●	●	●
	3	●	●	●	●	●	●
	4	●	●	●	●	●	●
	5	●	●	●	●	●	●
	6	●	●	●	●	●	●

There are 3 outcomes (of 36) that give a total of 4. This is the condition.

The other part of this question asks about the dice reading the same. There are 6 (of 36) of these in a top left to bottom right diagonal.

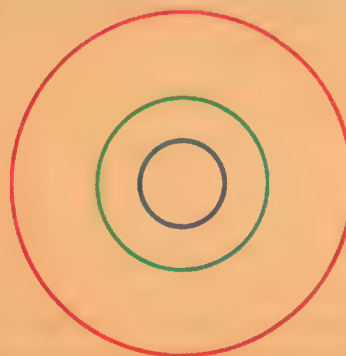
Using the conditional probability formula, we need the intersection of these two events: the total is 4 and the numbers are the same. There is 1 of 36 of these.

$$P(\text{numbers the same} | 4) = \frac{\frac{1}{36}}{\frac{3}{36}} = \frac{1}{3}$$

Note that the same result can be derived from the lattice diagram and considering the restricted domain (orange dots) One of the three of these has numbers the same.

Example D.5.3

Darts are thrown at random at the target shown:



The green circle has half the diameter of the red circle and the blue circle has half the diameter of the green circle.

R is the event that a dart lands inside the red circle, G that it lands within the green circle and B that it lands within the blue circle.

$P(R) = 1$, find:

- $P(G)$ and $P(B)$
- the probability that a dart lands inside the red circle but outside the green circle.
- the probability that a dart lands inside the red circle but outside the blue circle.
- the probability that a dart lands inside the blue circle given that it lands in the green circle.

- a The probabilities will be proportional to the area. The radius of the green circle is half that of the red circle. However, area is proportional to the square of the radius. The area of the green circle is one quarter (not one half) that of the red circle.

$$P(G) = \frac{1}{4} \text{ and } P(B) = \frac{1}{16}$$

- b The probability that a dart lands inside the red circle but outside the green circle is:

$$P(R \text{ not } G) = 1 - \frac{1}{4} = \frac{3}{4}$$

- c The probability that a dart lands inside the red circle but outside the blue circle is:

$$P(R \text{ not } B) = 1 - \frac{1}{16} = \frac{15}{16}$$

- d The probability that a dart lands inside the blue circle given that it lands in the green circle is:

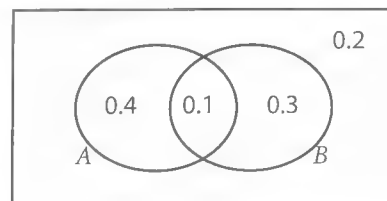
$$\begin{aligned} P(B|G) &= \frac{P(B \cap G)}{P(G)} \\ &= \frac{P(B)}{P(G)} \\ &= \frac{1}{16} \\ &= \frac{1}{4} \\ &= \frac{1}{4} \end{aligned}$$

Notice that we might have arrived at this conclusion by observing that the area of the blue circle is one quarter that of the green circle.

$$\begin{aligned} P(A \cap B) &= 0.5 + 0.4 - 0.8 \\ &= 0.1 \end{aligned}$$

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{0.1}{0.4} \\ &= 0.25 \end{aligned}$$

The Venn Diagram is:



b
$$\begin{aligned} P(B|A) &= \frac{P(B \cap A)}{P(A)} \\ &= \frac{0.1}{0.5} \\ &= 0.2 \end{aligned}$$

c
$$\begin{aligned} P(A'|B) &= \frac{P(A' \cap B)}{P(B)} \\ &= \frac{0.3}{0.4} \\ &= 0.75 \end{aligned}$$

Independence

Two events are independent if they do not affect one another. If we spin a fair coin, the probability of getting a head is 0.5. Suppose we do actually get a head. If we spin the coin again, the fact that we got a head the first time does not alter the probability that we get a second head.

Contrast this with the dealing of cards (without replacing them in the pack). Suppose the first card drawn is an ace. The chance of getting an ace is no longer $1/13$ as there are now only three aces in fifty one cards - so the probability of dealing an ace is now $3/51$.

There is a formal definition of independence:

Two events A and B are independent if and only if:

$$P(A|B) = P(A) \text{ and } P(B|A) = P(B)$$

Example D.5.4

Two events A and B are such that $P(A) = 0.5$, $P(B) = 0.4$ and $P(A \cup B) = 0.8$. Find:

a $P(A|B)$ b $P(B|A)$ c $P(A'|B)$.

a
$$P(A|B) = \frac{P(A \cap B)}{P(B)}, P(B) \neq 0$$

It is necessary to calculate: $P(A \cap B)$ using the addition rule:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

The consequence of this is:

Two events A and B are independent if, and only if:

$$P(A \cap B) = P(A) \times P(B)$$

These two conditions work both ways. If two events are independent, the conditions hold. If the conditions hold, the events are independent.

Do not confuse **independence** with events being **exclusive**. Exclusive events cannot occur together. A coin, for example, cannot simultaneously fall heads and tails.

As we have observed elsewhere, the principle of independence is important in improving the safety of mechanical devices.

Standard recreational scuba equipment consists of a single air tank. Connected to this are two separate 'regulators' (the breathing device that goes in the diver's mouth).



Having two separate breathing devices greatly increases safety. If one device fails, the diver switches to the other. Does this make the chance of losing the air supply from both devices independent. On the face of it, the answer to this looks like 'yes'.

However, it is actually 'no' as there is only one air tank. If the diver is careless and runs out of air, then both regulators will fail.

Some divers carry a small second air tank (known as a 'pony') and this restores independence.

Example D.5.5

The events A and B are such that: $P(A|B) = 0.2$, $P(B) = 0.5$ and $P(A \cup B) = 0.8$.

Are A and B independent?

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$\begin{aligned} P(A \cap B) &= P(A|B) \times P(B) \\ &= 0.2 \times 0.5 \\ &= 0.1 \end{aligned}$$

Next using the addition rule:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

It follows that:

$$\begin{aligned} P(A) &= P(A \cup B) - P(B) + P(A \cap B) \\ &= 0.8 - 0.5 + 0.1 \\ &= 0.4 \end{aligned}$$

The test for independence is that:

$$P(A \cap B) = P(A) \times P(B)$$

However: $0.1 \neq 0.4 \times 0.5$ so the events are dependent.

Example D.5.6

The events A and B are such that: $P(B|A) = 0.4$, $P(A) = 0.5$ and $P(B) = 0.3$.

Are A and B independent?

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

$$\begin{aligned} P(A \cap B) &= P(B|A) \times P(A) \\ &= 0.4 \times 0.5 \\ &= 0.2 \end{aligned}$$

Applying the test: $P(A \cap B) = P(A) \times P(B) \Rightarrow 0.2 = 0.4 \times 0.5$ the events are independent.

Example D.5.7

A card is drawn from a normal pack of 52 cards. Find the probability that the card is:

- a red b an ace

Are these events independent?

$$\begin{aligned} \text{a} \quad P(\text{red}) &= \frac{1}{2} \\ \text{b} \quad P(\text{ace}) &= \frac{4}{52} = \frac{1}{13} \end{aligned}$$

If the events are independent, $P(\text{red ace}) = P(\text{red}) \times P(\text{ace})$

$$P(\text{red ace}) = \frac{2}{52} = \frac{1}{26} \text{ and } P(\text{red}) \times P(\text{ace}) = \frac{1}{2} \times \frac{1}{13} = \frac{1}{26}$$

The events are dependent.

Exercise D.5.1

1. Two events A and B are such that:

$$P(A) = 0.3, P(B) = 0.5 \text{ and } P(A \cap B) = 0.1$$

Find:

- a $P(A \cup B)$
b $P(B')$
c $P(A' \cap B)$

2. A coin is tossed four times.

- a How many outcomes are there in the sample space?
b What is the probability of getting four heads?
c What is the probability of getting the same number of heads as tails?
d What is the probability of getting more heads than tails?

3. Two letters are randomly chosen, one after the other, from the word ARTICHOKE. What is the probability that these letters spell AT if:

- a the same letter can be chosen twice?
b the same letter cannot be chosen twice?

4. In a group of 80 students, 35 play baseball, 25 play football and 30 play neither.

- a Draw a Venn diagram to show these data.
b What is the probability that a randomly selected student plays both baseball and football?
c What is the probability that a randomly selected student plays baseball but not football?
d What is the probability that a randomly selected student plays neither sport?

5. Two events A and B are such that:

$$P(A) = 0.5, P(B) = 0.4 \text{ and } P(A \cap B) = 0.3$$

Find:

- a $P(A|B)$
b $P(B|A)$
c Are the events A and B independent?

6. Two cards are drawn from a normal pack without replacement.

- a What is the probability that the second card is an ace given that the first card was not an ace?
b Are the two draws independent?
c If we do not look at the first card, what is the probability that the second card is an ace?

7. Two letters are selected at random from the word ATTITUDE.

- a If the letters are selected with the first one selected being replaced before the second one is selected:
- i are the events independent?
 - ii what is the probability that both letters are Ts?
 - iii what is the probability that both letters are vowels?

b If the letters are selected without replacement:

- i are the events independent?
- ii what is the probability that both letters are Ts?
- iii what is the probability that both letters are vowels?

8. Given that $P(A) = 0.6$, $P(B) = 0.4$ and that A and B are independent events.

Find:

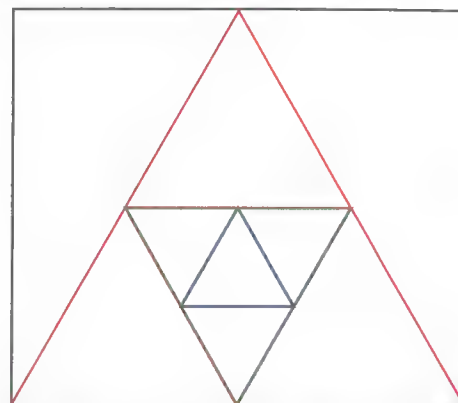
- a $P(A \cup B)$
- b $P(A \cap B)$
- c $P(A|B')$

9. 2 A group of 200 language students, all of whom speak at least one language have the following languages: Italian 99, Japanese 115, German 87, Italian and Japanese 44, German and Japanese 38 and Italian and German 31.

- a Draw a Venn diagram to show this information.
- b If a student is chosen at random, what is the probability that s/he speaks all three languages?
- c If a student is chosen at random, what is the probability that s/he speaks exactly two languages?
- d If a student is chosen at random, what is the probability that s/he speaks Italian given that she also speaks Japanese?
- e If two students are randomly chosen to represent the group on a student council, are

the selections independent of each other? Give a brief reason.

10. Points are chosen at random within the black rectangle.



Find the probabilities the the point lies:

- a within the red triangle.
- b within the green triangle.
- c within the blue triangle.
- d outside the red triangle.
- e inside the red triangle but outside the green triangle.
- f inside the red triangle but outside the blue triangle.
- g inside the green triangle given that the point is inside the red triangle.
- h inside the blue triangle given that the point is inside the red triangle.
- i inside the blue triangle given that the point is inside the green triangle.

Are the events:

'Falling inside the blue triangle' and 'falling inside the green triangle'.

Exclusive?

Independent?

Answers





Probability Theory has produced a number of paradoxes and seeming contradiction. It can be interesting to try to resolve these for yourself. We have appended some hints in the answers.

The Neglected Friend

Rodolfo has two friends that he likes to visit at the weekends.

Mimi can be reached by taking an eastbound train from his local station and Musetta by taking a westbound train. In order not to be seen to be favouring one of his friends, Rodolfo decides to take whichever train comes first after he has arrived at the station. Each train arrives at the station once per hour.

Never mind that this leaves him arriving unannounced!

After the arrangement has been in place for several months, Rodolfo discovers that he is almost always visiting Mimi and that he is neglecting Musetta.

What has gone wrong?

The Gamow-Stern Elevator Paradox

A high-rise building has one elevator. Harriet Highpoint works on a floor near the top of the building. She likes to take her lunch at a Bistro just down the street. She feels that most of the elevators that stop at her floor are heading up.

Lional Lowdown works on the second floor and likes to lunch in the rooftop Cafe. He feels that most of the elevators that stop at his floor are heading down.

Is this a real perception?

Would their complaints be resolved if there were multiple elevators?

Crown and Anchor

'Crown and Anchor' is a dice game that goes by a number of names (Chuck-a-Luck in the USA etc.). The game uses three cubic dice whose faces are marked 'spade', 'diamond', 'heart', 'club', 'crown' and 'anchor'.

The players bet on one of the symbols. The dice are rolled. If any of them show your symbol, you win. If not, you lose.

Players reason that each die gives them a $\frac{1}{6}$ th chance of winning. There are three dice so the game is fair.

It is not! Why?

The Wallet Paradox

Cassio and Iago agree to play a game. They will both place all the money in their wallet on the table. The person with the smallest amount wins all the money. Cassio, who has \$50, reasons that if he wins, he wins more than the \$50 he stands to win more than the \$50 he might lose. Iago, who has \$40, reasons in the same way. Both think the game is in their favour. Is it?

Utility

Utility means 'usefulness'. Most people agree that money is useful. However, \$1 does not have the same utility to everybody. An extra dollar to a millionaire will make little difference to his or her lifestyle and so has little utility. To a beggar it may mean the difference between life and death and therefore has enormous utility.

Charitable giving, in terms of utility at least, is a 'good action' in that it increases the total utility of money. There are some down sides to charity in that it may reduce a person's need to be self reliant. But that is another issue!

But what of gambling? If two people each have \$10, their money is of equal utility to both of them.

Next, they play a fair gambling game and each wagers \$1. One wins and now has a wealth of \$11. The other loses and now has a wealth of \$9. The dollar won now has less utility to the winner than the dollar lost has to the (poorer) loser. The total utility of the \$20 has been reduced - and the game was fair and nothing was lost to a casino. Gambling is a negative!

This rather gloomy account does not extend to money exchanged during commerce when goods are created.

D.6 Probability Distributions

SL 4.12

The main purpose of this chapter is to extend the work on the Normal Distribution that we undertook in Chapter D.6 of the Common Core Book. However, we will begin with a brief review of the ideas underlying probability distributions. These are, essentially, a complete list of all the outcomes that can happen along with the probabilities of each.

For example, if we spin three fair coins and count the number of heads obtained (calling this number X), there are four possible values for X (with $X = 0$ representing 3 tails).

The complete list of these outcomes is:

$$P(X = 0) = \frac{1}{8}$$

$$P(X = 1) = \frac{3}{8} \text{ (there are 3 places for the 1 head to occur)}$$

$$P(X = 2) = \frac{3}{8}$$

$$P(X = 3) = \frac{1}{8}$$

These probabilities add up to 1 - a feature common to all probability distributions as they list the probabilities of all possible outcomes.

This is the Binomial Distribution that we met before.

Whilst not specifically mentioned in the syllabus, students should realise that not all probability distributions have a finite number of outcomes.

Example D.6.1

A fair die is rolled. What is the probability that:

- the die rolls 6 on the first roll.
- the first six occurs on the fourth roll.
- the first six occurs on the n th roll.

Does this infinite distribution sum to 1?

- The die is fair so $P(1 \text{ roll for first } 6) = \frac{1}{6}$.
- This implies that there have been three non-sixes before the first six:

$$P(3 \text{ non-sixes, then six}) = \left(\frac{5}{6}\right)^3 \times \frac{1}{6} = \frac{125}{1296}$$

- If we define the variable X as being the number of rolls needed to obtain the first six, then, extending the reasoning of the previous part:

$$P(X = n) = \left(\frac{5}{6}\right)^{n-1} \times \frac{1}{6}$$

The first few values of this distribution are:

$$P(X = 1) = \frac{1}{6}$$

$$P(X = 2) = \left(\frac{5}{6}\right)^1 \times \frac{1}{6} = \frac{5}{36}$$

$$P(X = 3) = \left(\frac{5}{6}\right)^2 \times \frac{1}{6} = \frac{25}{36}$$

$$P(X = 4) = \left(\frac{5}{6}\right)^3 \times \frac{1}{6} = \frac{125}{1296}$$

and so on. Since this is an infinite list of numbers it is far from clear that it will sum to 1.

However, it is a convergent geometric series with $a = \frac{1}{6}$ and $r = \frac{5}{6}$

Recall that the sum of such a series is: $S_{\infty} = \frac{a}{1-r}$.

Review Chapter A.2 for a derivation of this.

In this case: $S_{\infty} = \frac{\frac{1}{6}}{1 - \frac{5}{6}} = \frac{\frac{1}{6}}{\frac{1}{6}} = 1$

This distribution is said to be **geometric** and thus fulfils this primary criterion necessary to qualify as a probability distribution.

The Normal Distribution

It is a good idea to remind yourself of the key difference between the normal distribution and the discrete distributions of the sort we have just been discussing.

This is that the variables are continuous. It is now meaningless to ask for the value of $P(X = 6)$.

The only questions that make sense are of the type 'What is $P(5.9 < X < 6.1)$?'.

This is because a measured quantity never takes an exact value. We can only say that it lies inside a finite range of error.

Another pitfall to avoid is the assumption that all continuous distributions are normal. The times you might expect to wait for a bus are not normally distributed. The Normal Distribution is, however, very commonly found as it refers to situations in which data deviates from a mean as a result of random factors.

We will begin by reviewing the basic skills of using a calculator to solve normal distribution problems.

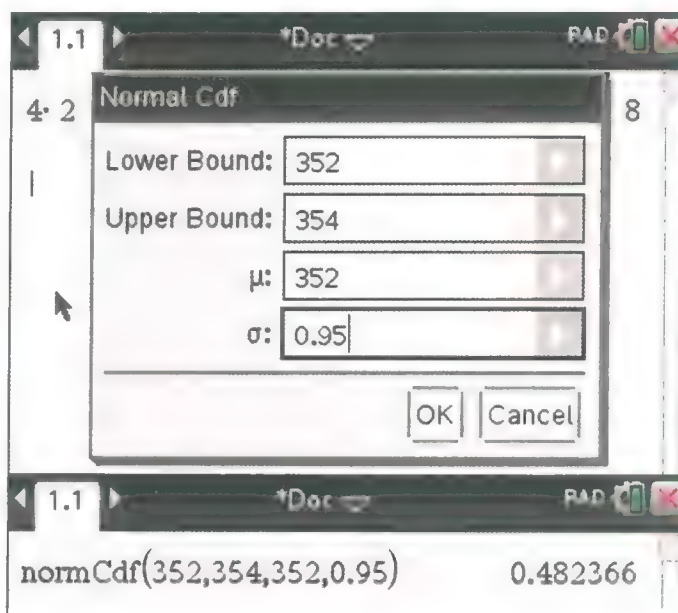
The first example reviews the basic techniques.

Example D.6.2

Iced Coffee bottles labelled 350 ml are filled automatically. A sample of 100 bottles was found to have a mean contents of 352 ml and a standard deviation of 0.95 ml. Find:

- The probability that a randomly chosen bottle contains between 352 ml and 354 ml.
- The percentage of bottles that contain less than the nominated amount.

- a Using a TI calculator:



$$P(352 < X < 354) \approx 0.48$$

- b This is $P(X < 350) \approx 0.0176$

Around 1.8% are under-filled.

We will now concentrate on inverse normal calculations including cases in which the mean and standard deviation are unknown.

An inverse calculation is the reverse of the above, as in the following example:

Example D.6.3

A group of army recruits have a mean height of 181 cm with a standard deviation of 2.6 cm. Find:

- a The height below which the shortest 15% of the recruits lie.
- b The height above which the tallest 25% of the recruits lie.

- a Using the Inverse Normal function with 0.15 representing the 15% of recruits.

$$\text{invNorm}(0.15, 181, 2.6) \quad 178.305$$

15% of the recruits are shorter than 178.3 cm.

- b $\text{invNorm}(0.75, 181, 2.6) \quad 182.754$

25% of the recruits are taller than 182.8 cm.

Exercise D.6.1

1. Find the value of the normally distributed variable X such that:

- a $P(X < x) = 0.841$ when $\mu = 4.4$ and $\sigma = 0.15$

- b $P(X < x) = 0.484$ when $\mu = 5.6$ and $\sigma = 2$

- c $P(X < x) = 0.517$ when $\mu = 12.9$ and $\sigma = 4.7$

- d $P(X < x) = 0.585$ when $\mu = -2$ and $\sigma = 0.56$

- e $P(X < x) = 0.023$ when $\mu = -4.6$ and $\sigma = 0.25$

2. Find the value of the normally distributed variable X such that:

- a $P(X > x) = 0.691$ when $\mu = 2.5$ and $\sigma = 1$

- b $P(X > x) = 0.826$ when $\mu = 34.5$ and $\sigma = 1.6$

- c $P(X > x) = 0.221$ when $\mu = 1.1$ and $\sigma = 0.65$

- d $P(X > x) = 0.773$ when $\mu = 0.5$ and $\sigma = 2$

- e $P(X > x) = 0.421$ when $\mu = -5$ and $\sigma = 2$

3. The weights of a group of 100 athletes are normally distributed with a mean of 65 kg and a standard deviation of 3.7 kg. The heaviest 10% of the group are selected for a special training program. What is the cut-off weight for entry to the program?

4. The times taken by a group of trainees to complete a competence task are normally distributed with a mean of 43 minutes and a standard deviation of 3 minutes 15 seconds. The slowest 15% of the trainees will fail the test. What is the pass/fail cut-off time?

5. The nominal power output of manufactured electric motors is 1.6 kW. A sample from the production line has a mean output of 1.71 kW with a standard deviation of 0.11 kW. The company policy is to stop the line and service it if more than 5% of the motors fall below the nominal output. Should the company stop the line?

6. A ball bearing manufacturer tests a sample from the production line. The sample has a mean diameter of 1.013 mm and a standard deviation of 0.0007 mm. If 1% are rejected for being too large, what is the largest acceptable ball bearing?

7. IQ tests scores are approximately normally distributed with a mean of 100 and a standard deviation of 10.

- a What proportion of the population lie in the range 70-100?

- b A sub-population has a mean IQ of 105. If students who score over 130 are classified as gifted, how many times more gifted students could be expected in the sub-population than in the general population.

- c Referring to the same sub-population as in part b:

What fraction of sub 70 students could be expected in the sub-population when compared with the wider population?

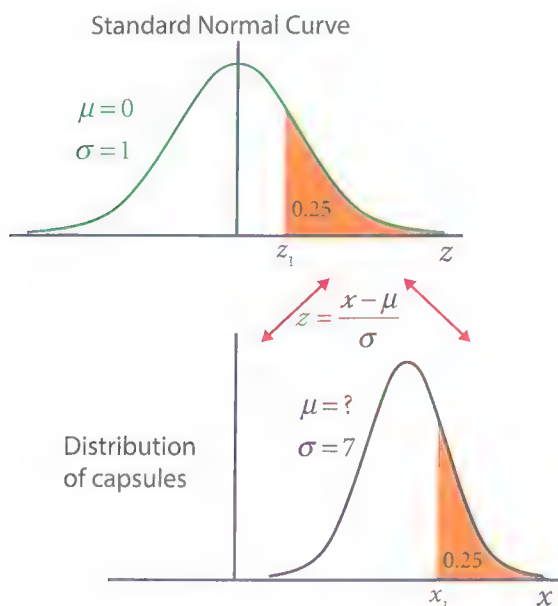
Unknown Mean and Standard Deviation

Example D.6.4

A quality control sample of drug capsules has a standard deviation of 7 mg. 25% of the capsules weigh more than 515 mg. The weights are normally distributed.

Find the mean weight of the sample.

The mean is unknown so we cannot use a calculator directly to solve this problem. The technique relies on relating the unknown distribution to the standard normal curve for which $\mu = 0$ and $\sigma = 1$.



First, we use a calculator to find z_1 , the variable value that gives a 0.25 probability that the variable is larger than z_1 and a 0.75 probability that the variable is smaller than z_1 .

$$\text{invNorm}(0.75, 0, 1) \quad 0.67449$$

So $z_1 = 0.67449$. This is now substituted into $z = \frac{x - \mu}{\sigma}$:

$$515 - \mu = 7 \times 0.67449$$

$$\mu = 515 + 7 \times 0.67449$$

$$= 519.72$$

Now that we have a mean, we can test the result using the parameters of the distribution. These are the data given:

$P(X > 515) = 0.25$ (or $P(X < 515) = 0.75$) and $\sigma = 7$ and our calculated mean: $\mu = 519.72$ (we will use this level of accuracy for the check:

$$\text{normCdf}(-\infty, 515, 519.72, 7) \quad 0.250065$$

The mean weight is 520 mg.

Example D.6.5

The lengths of fish caught by a trawler are normally distributed with a mean of 380 mm. Fish that are less than 350 mm are considered juveniles and are returned to the ocean. Of a catch of 452 fish, 27 are found to be juvenile. Find the standard deviation of the catch.

Find the mean weight of the sample.

Following the process used in the previous example, we first look at the standard normal curve.

27 out of 452 fish are juvenile so:

$$P(\text{juvenile}) = 27 \div 452 \approx 0.0597345$$

The Z value that gives this probability is:

$$\text{invNorm}(0.0597345, 0, 1) \quad -1.55701$$

Next, use the transformation $z = \frac{x - \mu}{\sigma}$.

The values are:

From the above calculation $z = -1.55701$

From the data: $\mu = 380$, $x = 350$.

$$-1.55701 = \frac{350 - 380}{\sigma}$$

$$-1.55701\sigma = 350 - 380$$

$$\sigma = \frac{-30}{-1.55701} \approx 19.27$$

Again, we will perform the check that the probability that a fish is juvenile (< 350 mm) $= 27 \div 452 \approx 0.0597345$ in a distribution for which $\mu = 380$ and $\sigma = 19.27$.

$$\text{normCdf}(-\infty, 350, 380, 19.27) \quad 0.059756$$

$$0.059756140112147 \cdot 452 \quad 27.0098$$

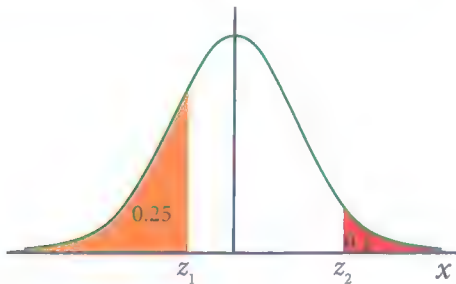
The second calculation ($\text{Ans} \times 452$) confirms that the predicted number of juveniles in a catch of 452 fish is 27.

Example D.6.6

A student makes 50 measurements of the time that it takes for the concentration of sulphuric acid in a reaction vessel to fall to half its initial value.

Assuming that the times are normally distributed and that one quarter of them are less than 235 seconds and 10% are more than 270 seconds, what is the mean time and the standard deviation of the times?

Using the Standard Normal Curve, the data is:



Using a calculator and remembering that the area to the left of the red area is 0.9:

$$\text{invNorm}(0.25, 0, 1) \quad -0.67449$$

$$\text{invNorm}(0.9, 0, 1) \quad 1.28155$$

$$z_1 = -0.67449 \text{ and } z_2 = 1.28155.$$

Next, we use the formula that represents the transformation from the 'true' distribution (X) to the standard normal curve (Z):

$$z = \frac{x - \mu}{\sigma}$$

For the left (orange) tail we have: $z_1 = -0.67449$ and $x_1 = 235$

$$z_1 = \frac{x_1 - \mu}{\sigma} \Rightarrow -0.67449 = \frac{235 - \mu}{\sigma} \Rightarrow -0.67449\sigma = 235 - \mu \quad [1]$$

For the left (red) tail we have: $z_2 = 1.28155$ and $x_2 = 270$

$$z_2 = \frac{x_2 - \mu}{\sigma} \Rightarrow 1.28155 = \frac{270 - \mu}{\sigma} \Rightarrow 1.28155\sigma = 270 - \mu \quad [2]$$

There are two simultaneous equations in μ and σ :

$$[2] - [1] \quad 1.28155\sigma - (-0.67449\sigma) = 270 - \mu - (235 - \mu)$$

$$1.95604\sigma = 35$$

$$\sigma = 17.89$$

Substituting in [2]:

$$1.28155 \times 17.89 = 270 - \mu$$

$$\mu = 247.07$$

The mean time is 247 sec and the standard deviation is 17.9 sec.

It is now possible to check that the answers fit the data:

$$\text{normCdf}(-\infty, 235, 247, 17.9) \quad 0.251304$$

$$\text{normCdf}(-\infty, 270, 247, 17.9) \quad 0.900589$$

Extension

Statisticians sometimes use the normal distribution to approximate other distributions that may be harder to handle from the computation standpoint.

In our previous calculations, we have been assuming a normal distribution. In each case this is an approximation. Until very recently, it was very difficult to answer a question such as:

If a fair die is rolled a thousand times, what is the probability that there are from 100 to 160 1s?

This is properly a binomial distribution with $n = 1\,000$ and $p = 1/6$.

Using the techniques developed in the Common Core:

$$\mu = np = 1\,000 \times 1/6 \approx 166.67$$

$$\sigma = \sqrt{np(1-p)} = \sqrt{1000 \times \frac{1}{6} \times \frac{5}{6}} \approx 11.785$$

There is also a subtle point here as the binomial distribution uses discrete data and the normal distribution uses continuous data.

If a discrete variable is in the range $[100, 160]$ then the random variable is in the range $(99.5, 160.5)$ - this is because 99.5 round to 100 etc.

Using the normal approximation:

$$P(99.5 < \text{sixes} < 160.5)$$

$$\text{normCdf}(99.5, 160.5, 166.67, 11.785) \quad 0.300297$$

Using an advanced statistical calculator:

$$\text{binomCdf}\left(1000, \frac{1}{6}, 100, 160\right) \quad 0.302802$$

These are not the same, but the normal approximation is much easier than the complete binomial calculation:

$$\begin{aligned} & \binom{1000}{100} \left(\frac{1}{6}\right)^{100} \left(\frac{5}{6}\right)^{900} + \binom{1000}{101} \left(\frac{1}{6}\right)^{101} \left(\frac{5}{6}\right)^{899} + \dots \\ & \dots + \binom{1000}{160} \left(\frac{1}{6}\right)^{160} \left(\frac{5}{6}\right)^{840} \end{aligned}$$

Exercise D.6.2

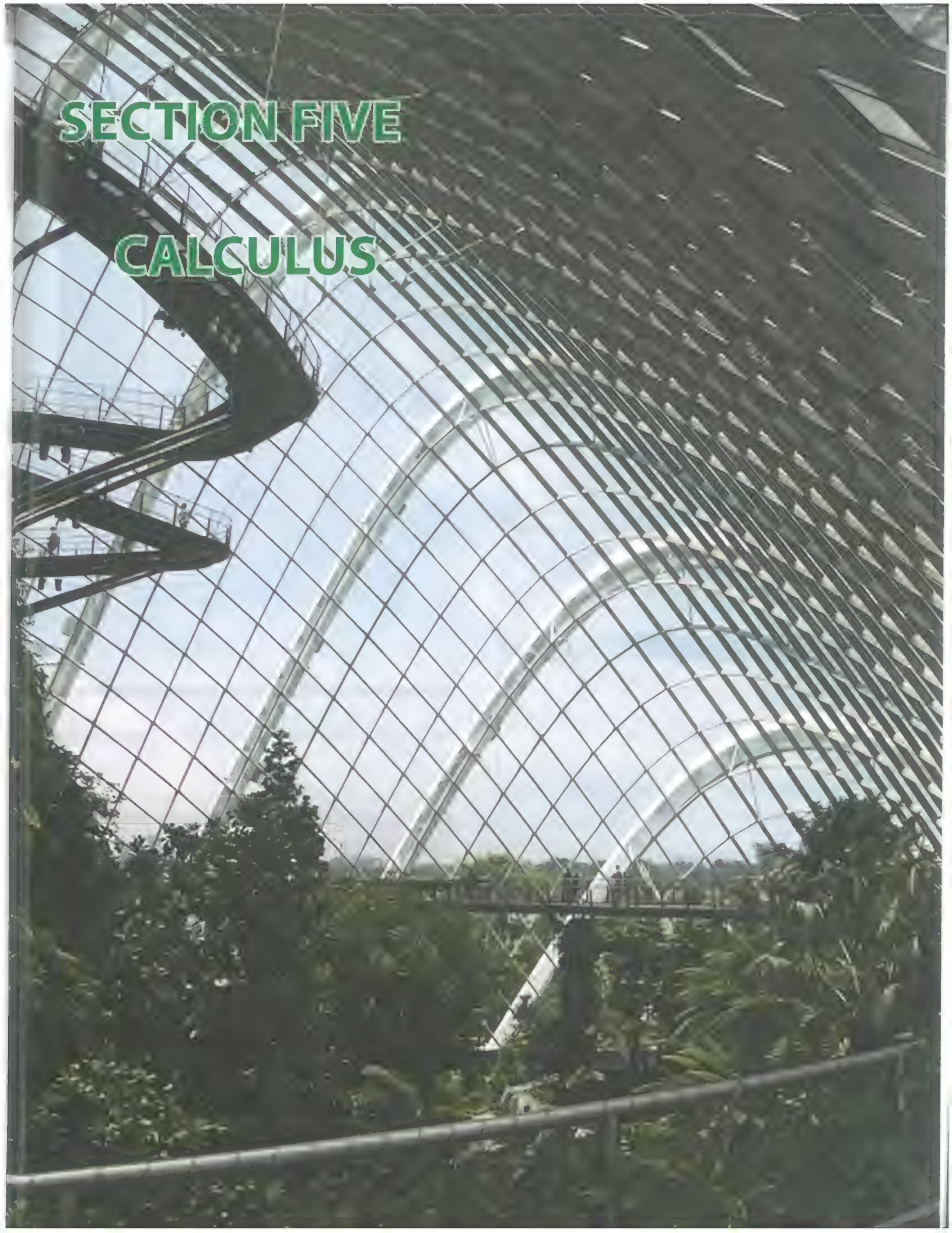
- For the following normally distributed variables X , find μ .
 - $P(X < 4.1) = 0.527, \sigma = 1.5$
 - $P(X < 41) = 0.779, \sigma = 5.2$
 - $P(X > 8.4) = 0.424, \sigma = 2.6$
 - $P(X < -1.4) = 0.474, \sigma = 3.1$
 - $P(X > 156) = 0.364, \sigma = 23$
- For the following normally distributed variables X , find σ .
 - $P(X < 5.3) = 0.115, \mu = 5.6$
 - $P(X > 4.75) = 0.977, \mu = 5.21$
 - $P(X < 125) = 0.338, \mu = 130$
 - $P(X < 297) = 0.591, \mu = 294$
 - $P(X > -3.35) = 0.462, \mu = -3.44$
- For the following normally distributed variables X , find μ and σ .
 - $P(X < 4) = 0.133$ and $P(X > 7) = 0.013$
 - $P(X < 3.9) = 0.023$ and $P(X > 4.8) = 0.055$
 - $P(X < 35) = 0.779$ and $P(X > 32) = 0.938$
 - $P(X < 170) = 0.345$ and $P(X > 177) = 0.159$
 - $P(X < -2.9) = 0.579$ and $P(X > -3.5) = 0.841$
- A sample of 100 eggs have a mean weight of 58 grams. 10 of the eggs weigh 55 grams or less. Assuming that the weights are normally distributed, find the standard deviation of the weights of the sample of eggs.
- A sample of 500 soft drink bottles were taken from a production line. 32 bottles in the sample contained 350.5 mL or less. 48 in the sample contain 357.0 mL or more. Assuming the contents are normally distributed:
 - Find the mean and standard deviation of the sample.
 - The number in the sample that might be expected to contain less than 349 mL.
- The times taken by a group of 200 trainees to complete a spot welding test are normally distributed. The fastest 36 students take 418 sec or less. 135 of the students take between 418 sec and 487 sec.
 - Find the mean and standard deviation of the sample.
 - Students who take longer than 490 sec have to repeat the tests. How many of these might you expect?
- In a road safety evaluation, the speeds of cars approaching an accident black-spot were measured. 37% of the cars were going at less than 53 kph and 9% were doing more than 63 kph. After the addition of safety signage, the study was repeated. 31% of the cars were going at less than 51 kph and 16% were doing more than 57 kph. Find the mean and standard deviations of the two sets and comment on the effectiveness of the signage.

Answers



SECTION FIVE

CALCULUS



E.2 Differentiation

SL 5.6

SL 5.7

Powers of the Variable

In Chapter E2 of the Common Core text, we looked at the problem of finding the gradient of curved functions. The approach was to use a limiting process and the result was a rule that we applied to integer powers of x in order to obtain the gradient function or derivative. This is a summary of the results and a reminder that there are two commonly used notations.

Notation	Function	Derivative
Newton	$f(x) = \dots$	$f'(x)$
Leibniz	$y = \dots$	$\frac{dy}{dx} = \dots$

The differentiation rule is:

If $f(x) = a \times x^n$ then $f'(x) = anx^{n-1}$ or

$$y = a \times x^n \text{ then } \frac{dy}{dx} = anx^{n-1}$$

Recall that we were able to differentiate polynomials term by term:

Example E.2.1

For each of these functions, find $\frac{dy}{dx}$:

a $y = 3x^2 - 4x + 1$

b $y = 3x - \frac{4}{x^2}$

a
$$\begin{aligned} \frac{dy}{dx} &= 3 \times 2 \times x^{2-1} - 4 \times 1 \times x^{1-1} + 0 \\ &= 6x - 4 \end{aligned}$$

b
$$\begin{aligned} \frac{dy}{dx} &= 3 \times 1 \times x^0 - 4 \times -2 \times x^{-2-1} \\ &= 3 + 8x^{-3} \\ &= 3 + \frac{8}{x^3} \end{aligned}$$

We now apply this rule to rational and, more generally, real exponents.

Example E.2.2

For each of these functions, find $\frac{dy}{dx}$:

a $y = 3\sqrt{x}$

b $y = \sqrt[3]{x} + 2x - 1$

c $y = \frac{4}{\sqrt{x}} + \frac{2}{x}$

d $y = x^\pi$

e $y = \pi^2$

$$\begin{aligned} \text{a} \quad y &= 3\sqrt{x} = 3x^{1/2} \\ \frac{dy}{dx} &= 3 \times \frac{1}{2} \times x^{1/2-1} \\ &= \frac{3}{2} x^{-1/2} \\ &= \frac{3}{2\sqrt{x}} \end{aligned}$$

$$\begin{aligned} \text{b} \quad y &= \sqrt[3]{x} + 2x - 1 = x^{1/3} + 2x - 1 \\ \frac{dy}{dx} &= \frac{1}{3} \times x^{1/3-1} + 2 \\ &= \frac{1}{3} \times x^{-2/3} + 2 \\ &= \frac{1}{3x^{2/3}} + 2 \end{aligned}$$

$$\begin{aligned} \text{c} \quad y &= \frac{4}{\sqrt{x}} + \frac{2}{x} = 4x^{-1/2} + 2x^{-1} \\ \frac{dy}{dx} &= 4 \times \left(-\frac{1}{2}\right) x^{-1/2-1} + 2 \times (-1) x^{-1-1} \\ &= -2x^{-3/2} - 2x^{-2} \\ &= -\frac{2}{\sqrt{x^3}} - \frac{2}{x^2} \end{aligned}$$

$$\begin{aligned} \text{d} \quad y &= x^\pi \\ \frac{dy}{dx} &= \pi x^{\pi-1} \end{aligned}$$

π is a constant and must, effectively be treated as if it were just an ordinary number such as 2.

$$\begin{aligned} \text{e} \quad y &= \pi^2 \\ \frac{dy}{dx} &= 0 \end{aligned}$$

Again, π is a constant and differentiates to zero.

$$\text{g} \quad 2\sqrt{x} - \frac{3}{x} + 12 \quad \text{h} \quad x\sqrt{x} + \frac{1}{\sqrt{x}} + 2$$

$$\text{i} \quad 5\sqrt[3]{x^2} - 9x \quad \text{j} \quad 5x - \frac{x}{\sqrt{x}} + \frac{4}{5x^2}$$

$$\text{k} \quad 8\sqrt{x} + 3x^{-5} + \frac{x}{2} \quad \text{l} \quad \frac{x}{\sqrt{x^3}} - \frac{2}{x}\sqrt{x^3} + \frac{1}{3}x^3$$

2. Find the derivative of each of the following.

$$\text{a} \quad \sqrt{x}(x+2) \quad \text{b} \quad (x+1)(x^3-1)$$

$$\text{c} \quad x\left(x^2 + 1 - \frac{1}{x}\right), x \neq 0 \quad \text{d} \quad \frac{2x-1}{x}, x \neq 0$$

$$\text{e} \quad \frac{\sqrt{x}-2}{\sqrt{x}}, x > 0 \quad \text{f} \quad \frac{x^2-x+\sqrt{x}}{2x}, x \neq 0$$

$$\text{g} \quad \frac{3x^2-7x^3}{x^2}, x \neq 0 \quad \text{h} \quad \left(x - \frac{2}{x}\right)^2, x \neq 0$$

$$\text{i} \quad \left(x + \frac{1}{x^2}\right)^2, x \neq 0 \quad \text{j} \quad \sqrt{3x} - \frac{1}{3\sqrt{x}}, x > 0$$

$$\text{k} \quad (x-5\sqrt{x})^2, x \geq 0 \quad \text{l} \quad \left(\frac{1}{\sqrt{x}} - \sqrt{x}\right)^3, x > 0$$

$$\text{a} \quad \text{Show that if } f(x) = x^2 - x, \text{ then } f'(x) = 1 + \frac{2f(x)}{x}.$$

$$\begin{aligned} \text{b} \quad &\text{Show that if } f(x) = \sqrt{2x} - 2\sqrt{x}, x \geq 0, \\ &\text{then } \sqrt{2x}f'(x) = 1 - \sqrt{2}, x > 0 \end{aligned}$$

$$\begin{aligned} \text{c} \quad &\text{Show that if } y = ax^n \text{ where } a \text{ is real and} \\ &n \in \mathbb{N}, \text{ then } \frac{dy}{dx} = \frac{ny}{x}, x \neq 0 \end{aligned}$$

$$\text{d} \quad \text{Show that if } y = \frac{1}{\sqrt{x}}, x > 0, \text{ then } \frac{dy}{dx} + \frac{y}{2x} = 0.$$

Exercise E.2.1

1. Find the derivative of each of the following.

$$\text{a} \quad \frac{1}{x^3} \quad \text{b} \quad \sqrt{x^3} \quad \text{c} \quad \sqrt{x^5}$$

$$\text{d} \quad \sqrt[3]{x} \quad \text{e} \quad 4\sqrt{x} \quad \text{f} \quad 6\sqrt{x^3}$$

Differentiating with variables other than x and y

Although it was convenient to establish the underlying theory of differentiation based on the use of the variables x and y , it must be pointed out that not all expressions are written in terms of x and y . In fact, many of the formulae that we use are written in terms of variables other than y and x , e.g. volume, V , of a sphere is given by $V = \frac{4}{3}\pi r^3$, where r is its radius. The displacement of a particle moving with constant acceleration is given by $s = ut + \frac{1}{2}at^2$. However, it is reassuring to know that the rules are the same regardless of the variables involved. Thus, if we have that y is a function of x , we can differentiate y with respect to (w.r.t.) x to find $\frac{dy}{dx}$.

On the other hand, if we have that y is a function of t , we would differentiate y w.r.t. t and write $\frac{dy}{dt}$.

Similarly, if W was a function of θ , we would differentiate W w.r.t. θ and write $\frac{dW}{d\theta}$.

Example E.2.3

Differentiate the following with respect to the appropriate variable.

a $V = \frac{4}{3}\pi r^3$

b $p = 3w^3 - 2w + 20$

c $s = 10t + 4t^2$

- a As V is a function of r , we need to differentiate V with respect to r :

$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = \frac{4}{3}\pi(3r^2) = 4\pi r^2.$$

- b This time p is a function of w , and so we would differentiate p with respect to w :

$$p = 3w^3 - 2w + 20 \Rightarrow \frac{dp}{dw} = 9w^2 - 2.$$

- c In this expression we have that s is a function of t and so we differentiate s w.r.t. t :

$$s = 10t + 4t^2 \Rightarrow \frac{ds}{dt} = 10 + 8t.$$

Exercise E.2.2

1. Differentiate the following functions with respect to the appropriate variable.

a $s = 12t^4 - \sqrt{t}$ b $Q = \left(n + \frac{1}{n^2}\right)^2$

c $P = \sqrt{r}(r + \sqrt[3]{r} - 2)$ d $T = \frac{(\theta - \sqrt{\theta})^3}{\theta}$

e $A = 40L - L^3$ f $F = \frac{50}{v^2} - v$

g $V = 2l^3 + 5l$ h $A = 2\pi h + 4h^2$

2. Differentiate the following with respect to the independent variable:

a $v = \frac{2}{3}\left(5 - \frac{2}{t^2}\right)$ b $S = \pi r^2 + \frac{20}{r}$

c $q = \sqrt{s^5} - \frac{3}{s}$ d $h = \frac{2 - t + t^2}{t^3}$

e $L = \frac{4 - \sqrt{b}}{b}$

f $W = (m - 2)^2(m + 2)$

Derivatives of Transcendental Functions

A transcendental function is a function that cannot be constructed in a finite number of steps from elementary functions and their inverses. Some examples of these functions are $\sin x$, $\cos x$, $\tan x$, the exponential function, e^x , and the logarithmic function $\log_e x$ (or $\ln x$).

Also in this section we look at derivatives of expressions that involve the **product** of two functions, the **quotient** of two functions and the **composite** of two functions. Each of these types of expressions will lead to some standard rules of differentiation.

Derivative of circular trigonometric functions

We begin by considering the trigonometric functions, i.e. the sine, cosine and tangent functions.

There are a number of approaches that can be taken to achieve our goal. In this instance we will use two different approaches to find the derivative of the function $x \mapsto \sin(x)$.

Letting $f(x) = \sin(x)$ and using the definition from first principles we have:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h} \end{aligned}$$

From the identities discussed in Chapter C.5:

$$\frac{\sin(x+h) - \sin(x)}{h} = \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

We are taking the limit as h becomes small.

$h \rightarrow 0, \cos(h) \rightarrow 1$ so:

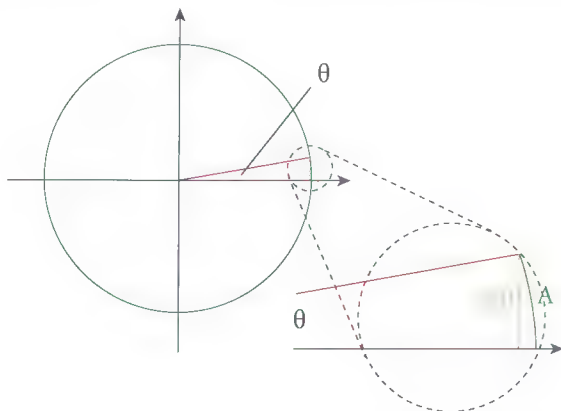
$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \left(\frac{\sin(x) \times 1 + \cos(x)\sin(h) - \sin(x)}{h} \right) \\ &= \lim_{h \rightarrow 0} \left(\frac{\cos(x)\sin(h)}{h} \right) \end{aligned}$$

The result of this limit is not immediately obvious.

$$\text{It simplifies to: } f'(x) = \lim_{h \rightarrow 0} \left(\frac{\sin(h)}{h} \right) \times \cos(x)$$

This depends on the result of the limit: $\lim_{h \rightarrow 0} \left(\frac{\sin(h)}{h} \right)$

The result of this is suggested (not proved) by the unit circle.



From the definition of angle, the arc length (green) is equal to the angle in radians, θ . The light green line is equal to $\sin \theta$. The smaller the angle becomes, the closer these to quantities become to one another.

This strongly suggests that: $\lim_{h \rightarrow 0} \left(\frac{\sin(h)}{h} \right) = 1$ and hence:

$$f(x) = \sin(x) \Rightarrow f'(x) = \cos(x)$$

This is only true if the angle is measured in radians - a very good reason for using radians rather than degrees!

Similarly, we have:

$$f(x) = \cos(x) \text{ then } f'(x) = -\sin(x)$$

$$f(x) = \tan(x) \text{ then } f'(x) = \sec^2(x)$$

Derivative of the exponential function

Consider the exponential function $x \mapsto e^x$. Although we could apply the same graphical method which was used to determine the derivative of the sine function, this time we shall make use of the definition of the derivative, i.e.

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h} \\ &= \lim_{h \rightarrow 0} \frac{e^x e^h - e^x}{h} \text{ using } e^x e^h = e^{x+h} \\ &= \lim_{h \rightarrow 0} \frac{e^x (e^h - 1)}{h} \\ &= e^x \lim_{h \rightarrow 0} \frac{(e^h - 1)}{h} \text{ Use a numerical method to find that this limit} = 1 \\ &= e^x \times 1 \\ &= e^x \end{aligned}$$

Notice that this function has a derivative that is the same as itself.

Derivative of the natural log function

Consider the function $x \mapsto \log_e x$. As in the previous case, we use the definition of the derivative to establish the gradient function of $x \mapsto \log_e x$.

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\log_e(x+h) - \log_e(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\log_e \left(\frac{x+h}{x} \right)}{h} \text{ (Using the log laws)} \end{aligned}$$

The next step is a little tricky, so we write it down first and then see how we arrive at the result.

$$= \frac{1}{x} \lim_{h \rightarrow 0} \log_e \left(1 + \frac{h}{x} \right)^{\frac{x}{h}}$$

To get to this step we proceed as follows:

$$\frac{\log_e\left(1+\frac{h}{x}\right)}{h} = \frac{1}{x} \cdot x \frac{\log_e\left(1+\frac{h}{x}\right)}{h} = \frac{1}{x} \cdot \left(\frac{x}{h}\right) \log_e\left(1+\frac{h}{x}\right) \\ = \frac{1}{x} \cdot \log_e\left(1+\frac{h}{x}\right)^{\frac{x}{h}}$$

Then, as the argument in the limit is h (i.e. it is independent of x) we have:

$$\lim_{h \rightarrow 0} \frac{1}{x} \cdot \log_e\left(1+\frac{h}{x}\right)^{\frac{x}{h}} = \frac{1}{x} \lim_{h \rightarrow 0} \log_e\left(1+\frac{h}{x}\right)^{\frac{x}{h}}$$

Then, as the log function is a continuous function, we have that the limit of the log is the same as the log of the limit so that

$$\lim_{h \rightarrow 0} \log_e\left(1+\frac{h}{x}\right)^{\frac{x}{h}} = \log_e\left[\lim_{h \rightarrow 0} \left(1+\frac{h}{x}\right)^{\frac{x}{h}}\right]$$

However, we also have that $\lim_{h \rightarrow 0} \left(1+\frac{h}{x}\right)^{\frac{x}{h}} = e$ and so we end

up with the result that

$$\frac{1}{x} \lim_{h \rightarrow 0} \log_e\left(1+\frac{h}{x}\right)^{\frac{x}{h}} = \frac{1}{x} \log_e e = \frac{1}{x} \times 1 = \frac{1}{x}$$

And so, we have that if $f(x) = \log_e x$ then $f'(x) = \frac{1}{x}$

Derivative of a Product of Functions

Many functions can be written as the product of two (or more) functions. For example, the function $y = (x^3 - 2x)(x^2 + x - 3)$ is made up of the product of two simpler functions of x . In fact, expressions such as these take on the general form $y = u \times v$ or $y = f(x) \times g(x)$ where (in this case) we have $u = f(x) = (x^3 - 2x)$ and $v = g(x) = (x^2 + x - 3)$.

To differentiate such expressions we use the product rule, which can be written as:

Function	Derivative
$y = u \times v$	$\frac{dy}{dx} = \frac{du}{dx} \times v + u \times \frac{dv}{dx}$
$y = f(x) \times g(x)$	$\frac{dy}{dx} = f'(x) \times g(x) + f(x) \times g'(x)$

Example E.2.4

Differentiate the following.

- a $x^2 \sin(x)$ b $(x^3 - 2x + 1)e^x$
c $\frac{1}{x} \log_e(x)$

- a A useful method to find the derivative of a product makes use of the following table:

Function	Derivative
$u = x^2$	$\frac{du}{dx} = 2x$
$v = \sin(x)$	$\frac{dv}{dx} = \cos(x)$

$2x \sin(x)$
 $x^2 \cos(x)$
 Adding
 $2x \sin(x) + x^2 \cos(x)$

Let $y = x^2 \sin(x)$ so that $u = x^2$ and $v = \sin(x)$.

So that $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = \cos(x)$.

Using the product rule we have:

$$\begin{aligned} \frac{dy}{dx} &= \frac{du}{dx} \times v + u \times \frac{dv}{dx} \\ &= 2x \times \sin(x) + x^2 \times \cos(x) \\ &= 2x \sin(x) + x^2 \cos(x) \end{aligned}$$

- b Let

$y = (x^3 - 2x + 1)e^x$ so that $u = (x^3 - 2x + 1)$ and $v = e^x$.

Then, $\frac{du}{dx} = 3x^2 - 2$ and $\frac{dv}{dx} = e^x$.

Using the product rule:

$$\begin{aligned} \frac{dy}{dx} &= \frac{du}{dx} \times v + u \times \frac{dv}{dx} \\ &= (3x^2 - 2) \times e^x + (x^3 - 2x + 1) \times e^x \\ &= (3x^2 - 2 + x^3 - 2x + 1)e^x \\ &= (x^3 + 3x^2 - 2x - 1)e^x \end{aligned}$$

c This time set up a table:

Let $y = \frac{1}{x} \log_e x$ with $u = \frac{1}{x}$ and $v = \log_e x$.

Function	Derivative
$u = \frac{1}{x}$	$\frac{du}{dx} = -\frac{1}{x^2}$
$v = \log_e x$	$\frac{dv}{dx} = \frac{1}{x}$

$$\begin{aligned} \text{Adding: } \frac{dy}{dx} &= \frac{1}{x^2} \times \log_e x + \frac{1}{x} \times \frac{1}{x} \\ &= -\frac{1}{x^2} \times \log_e x + \frac{1}{x^2} \\ &= \frac{1}{x^2} (1 - \log_e x) \end{aligned}$$

Derivative of a Quotient of Functions

In the same way as we have a rule for the product of functions, we also have a rule for the quotient of functions.

For example, the function $y = \frac{x^2}{x^3 + x - 1}$

is made up of two simpler functions of x . Expressions like this take on the general form

$$y = \frac{u}{v} \quad \text{or} \quad y = \frac{f(x)}{g(x)}.$$

For the example shown above, we have that $u = x^2$ and $v = x^3 + x - 1$.

As for the product rule, we state the result.

$$y = u(x) \times v(x) \Rightarrow \frac{dy}{dx} = u(x) \times v'(x) + u'(x) \times v(x)$$

To differentiate such expressions we use the quotient rule, which can be written as:

Function	Derivative
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{\frac{du}{dx} \times v - u \times \frac{dv}{dx}}{v^2}$
$y = \frac{f(x)}{g(x)}$	$\frac{dy}{dx} = \frac{f'(x) \times g(x) - f(x) \times g'(x)}{[g(x)]^2}$

Example E.2.5

Differentiate the following.

a $\frac{x^2 + 1}{\sin(x)}$ b $\frac{e^x + x}{x + 1}$ c $\frac{\sin(x)}{1 - \cos(x)}$

a We express $\frac{x^2 + 1}{\sin(x)}$ in the form $y = \frac{u}{v}$, so that:

$$u = x^2 + 1 \quad \text{and} \quad v = \sin(x).$$

Giving the following derivatives, $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = \cos(x)$

Using the quotient rule we have,

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{du}{dx} \times v - u \times \frac{dv}{dx}}{v^2} \\ &= \frac{2x \times \sin(x) - (x^2 + 1) \times \cos(x)}{[\sin(x)]^2} \\ &= \frac{2x \sin(x) - (x^2 + 1) \cos(x)}{\sin^2(x)} \end{aligned}$$

b First express $\frac{e^x + x}{x + 1}$ in the form $y = \frac{u}{v}$.

so that $u = e^x + x$ and $v = x + 1$ and

$\frac{du}{dx} = e^x + 1$ and $\frac{dv}{dx} = 1$. Using the quotient rule, we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{du}{dx} \times v - u \times \frac{dv}{dx}}{v^2} = \frac{(e^x + 1) \times (x + 1) - (e^x + x) \times 1}{(x + 1)^2} \\ &= \frac{xe^x + e^x + x + 1 - e^x - x}{(x + 1)^2} \\ &= \frac{xe^x + 1}{(x + 1)^2} \end{aligned}$$

c Express the quotient $\frac{\sin(x)}{1 - \cos(x)}$ in the form $y = \frac{u}{v}$,

so that $u = \sin(x)$ and $v = 1 - \cos(x)$.

Then $\frac{du}{dx} = \cos(x)$ and $\frac{dv}{dx} = \sin(x)$.

Using the quotient rule, we have

$$\frac{dy}{dx} = \frac{\frac{du}{dx} \times v - u \times \frac{dv}{dx}}{v^2} = \frac{\cos(x) \times (1 - \cos(x)) - \sin(x) \times \sin(x)}{(1 - \cos(x))^2}$$

$$\begin{aligned}
 &= \frac{\cos(x) - \cos^2(x) - \sin^2(x)}{(1 - \cos(x))^2} \\
 &= \frac{\cos(x) - (\cos^2(x) + \sin^2(x))}{(1 - \cos(x))^2} \\
 &= \frac{\cos(x) - 1}{(1 - \cos(x))^2} \\
 &= -\frac{(1 - \cos(x))}{(1 - \cos(x))^2} \\
 &= -\frac{1}{(1 - \cos(x))}
 \end{aligned}$$

The Chain Rule

To find the derivative of $x^3 + 1$ we let $y = x^3 + 1$ so that $\frac{dy}{dx} = 3x^2$.

Next consider the derivative of the function $y = (x^3 + 1)^2$. We first expand the brackets, $y = x^6 + 2x^3 + 1$, and obtain $\frac{dy}{dx} = 6x^5 + 6x^2$.

This expression can be simplified (i.e. factorised), giving:

$$\frac{dy}{dx} = 6x^2(x^3 + 1).$$

In fact, it isn't too great a task to differentiate the function $y = (x^3 + 1)^3$.

As before, we expand $y = x^9 + 3x^6 + 3x^3 + 1$

$$\text{so that } \frac{dy}{dx} = 9x^8 + 18x^5 + 9x^2.$$

Factorizing this expression we now have:

$$\frac{dy}{dx} = 9x^2(x^6 + 2x^3 + 1) = 9x^2(x^3 + 1)^2.$$

But what happens if we need to differentiate the expression $y = (x^3 + 1)^8$? We could expand and obtain a polynomial with 9 terms (!), which we then proceed to differentiate and obtain a polynomial with 8 terms ... and of course, we can then factorise that polynomial. The question arises, "Is there an easier way to do this?"

We can obtain some idea of how to do this by summarizing the results found so far:

Function	Derivative	(Factored form)
$y = x^3 + 1$	$\frac{dy}{dx} = 3x^2$	$3x^2$
$y = (x^3 + 1)^2$	$\frac{dy}{dx} = 6x^5 + 6x^2$	$2 \times 3x^2(x^3 + 1)$
$y = (x^3 + 1)^3$	$\frac{dy}{dx} = 9x^8 + 18x^5 + 9x^2$	$3 \times 3x^2(x^3 + 1)^2$

Continuing $y = (x^3 + 1)^4$,

$$\frac{dy}{dx} = 12x^{11} + 36x^8 + 36x^5 + 12x^2 \text{ or } 4 \times 3x^2(x^3 + 1)^3$$

The pattern that is emerging is that if:

$$y = (x^3 + 1)^n \text{ then } \frac{dy}{dx} = n \times 3x^2(x^3 + 1)^{n-1}.$$

In fact, if we consider the term inside the brackets as one function, so that the expression is actually a composition of two functions, namely that of $x^3 + 1$ and the power function, we can write $u = x^3 + 1$ and $y = u^n$.

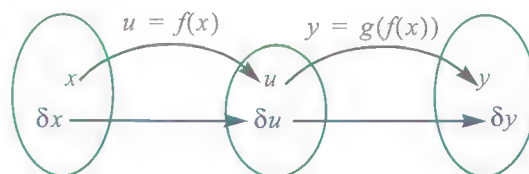
So that $\frac{dy}{du} = nu^{n-1} = n(x^3 + 1)^{n-1}$ and $\frac{du}{dx} = 3x^2$, giving

the result:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Is this a 'one-off' result, or can we determine a general result that will always work?

To explore this we use a graphical approach to see why it may be possible to obtain a general result.



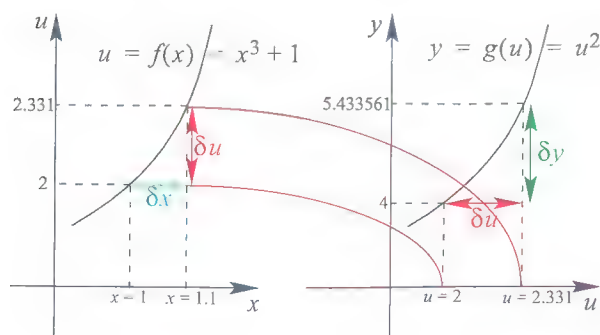
We start by using the above example and then move onto a more general case. For the function $y = (x^3 + 1)^2$, we let $u = x^3 + 1$ ($= g(x)$) and so $y = u^2$ ($= f(g(x))$). We need to find what effect a small change in x will have on the function y (via u), i.e. what effect will δx have on y ?

We have a sort of **chain reaction**, that is, a small change in x , δx , will produce a change in u , δu , which in turn will produce a change in y , δy ! It is the path from δx to δy that we are interested in.

This can be seen when we produce a graphical representation of the discussion so far.

We start by looking at the effect that a change in x has on u :

Then we observe the effect that the change in u has on y :



We then have $\delta x = 1.1 - 1 = 0.1$ and $\delta u = 2.331 - 2 = 0.331$

Similarly, $\delta u = 2.331 - 2 = 0.331$ and $\delta y = 5.433561 - 4 = 1.433561$

Based on these results, the following relationship can be seen to hold:

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}$$

The basic outline in proving this result is shown in the following argument:

Let δx be a small increment in the variable x and let δu be the corresponding increment in the variable u . This change in u will in turn produce a corresponding change δy in y .

As δx tends to zero, so does δu . We will assume that $\delta u \neq 0$ when $\delta x \neq 0$.

Hence we have that $\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \cdot \frac{\delta u}{\delta x} \Rightarrow \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta u} \cdot \frac{\delta u}{\delta x}$

$$= \left(\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta u} \right) \cdot \left(\lim_{\delta x \rightarrow 0} \frac{\delta u}{\delta x} \right) \quad \text{Given that:}$$

$$= \left(\lim_{\delta u \rightarrow 0} \frac{\delta y}{\delta u} \right) \cdot \left(\lim_{\delta x \rightarrow 0} \frac{\delta u}{\delta x} \right) \quad \delta x \rightarrow 0 \Rightarrow \delta u \rightarrow 0$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

We then have the result: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Using the chain rule

We will work our way through an example, showing the critical steps involved when using the chain rule.

This is demonstrated by finding the derivative of the function $y = \sin(x^2)$.

Step 1	Recognition
This is the most important step when deciding if using the chain rule is appropriate. In this case we <i>recognise</i> that the function $y = \sin(x^2)$ is a composite of the sine and the squared functions.	
Step 2	Define u (or $g(x)$)
Let the 'inside' function be u . In this case, we have that $u = x^2$	
Step 3	Differentiate u (with respect to x)
$\frac{du}{dx} = 2x$	
Step 4	Express y in terms of u
$y = \sin(u)$	
Step 5	Differentiate y (with respect to u)
$\frac{dy}{du} = \cos(u)$	
Step 6	Use the chain rule
$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos(u) \times 2x = 2x \cos(x^2)$	

Example E.2.6

Differentiate the following functions.

a $y = \log_e(x + \cos x)$

b $f(x) = (1 - 3x^2)^4$

a Begin by letting $u = x + \cos(x) \Rightarrow \frac{du}{dx} = 1 - \sin(x)$.

Express y in terms of u , that is,

$$y = \log_e u \Rightarrow \frac{dy}{du} = \frac{1}{u} \left(= \frac{1}{x + \cos(x)} \right).$$

Using the chain rule we have:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{x + \cos(x)} \cdot (1 - \sin(x)) = \frac{1 - \sin(x)}{x + \cos(x)}$$

b This time we let $g(x) = 1 - 3x^2$, so that $g'(x) = -6x$.

Now let $f(x) = (h \circ g)(x)$ so that $h(g(x)) = (g(x))^4$ and $h'(g(x)) = 4(g(x))^3$.

Therefore, using the chain rule we have

$$\begin{aligned} f'(x) &= (h \circ g)'(x) = h'(g(x)) \cdot g'(x) \\ &= 4(g(x))^3 \times (-6x) \\ &= -24x(1 - 3x^2)^3 \end{aligned}$$

Some standard derivatives

Often we wish to differentiate expressions of the form $y = \sin(2x)$ or $y = e^{5x}$ or other such functions, where the x term only differs by a constant factor from that of the basic function. That is, the only difference between $y = \sin(2x)$ and $y = \sin(x)$ is the factor '2'. We can use the chain rule to differentiate such expressions:

Let $u = 2x$, giving $y = \sin(u)$ and so

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos(u) \times 2 = 2\cos(2x)$$

Similarly,

Let $u = 5x$, giving $y = e^u$ and so

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u \times 5 = 5e^{5x}$$

Because of the nature of such derivatives, functions such as these form part of a set of functions that can be considered as having derivatives that are often referred to as standard derivatives. Although we could make use of the chain rule to differentiate these functions, they should be viewed as standard derivatives.

These standard derivatives are shown in the table (where k is some real constant):

y	$\frac{dy}{dx}$
$\sin(kx)$	$k\cos(kx)$
$\cos(kx)$	$-k\sin(kx)$
$\tan(kx)$	$k\sec^2(kx)$
e^{kx}	ke^{kx}
$\log_e(kx)$	$\frac{1}{x}$

Notice, the only derivative that does not involve the constant k is that of the logarithmic function. This is because letting $u = kx$, we have $y = \log(u)$ so:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \times k = \frac{1}{kx} \times k = \frac{1}{x}.$$

When should the chain rule be used?

A good **first rule** to follow is: If the expression is made up of a pair of brackets and a power, then the chances are that you will need to use the chain rule.

As a start, the expressions in the list of results that follows would require the use of the chain rule. Notice then that in each case the expression can be (or already is) written in 'power form'. That is, of the form $y = [f(x)]^n$.

Some examples to be on the lookout for:

$$y = (2x + 6)^5 \text{ - Let } u = 2x + 6 \text{ and } y = u^5$$

$$y = \sqrt{(2x^3 + 1)} \text{ - Let } u = 2x^3 + 1 \text{ and } y = u^{\frac{1}{2}}$$

$$y = \frac{3}{(x-1)^2}, x \neq 1 \text{ - Let } u = x-1 \text{ and } y = 3u^{-2}$$

$$y = \frac{1}{\sqrt[3]{e^{-x} + e^x}} \text{ - Let } u = e^{-x} + e^x \text{ and } y = u^{-\frac{1}{3}}$$

We now look at some of the more demanding derivatives which combine at least two rules of differentiation, for example, the need to use both the quotient rule and the chain rule, or the product rule and the chain rule.

Example E.2.7

Differentiate the following.

$$a \quad y = \sqrt{1 + \sin^2 x}$$

$$b \quad y = e^{\frac{1}{3}\sin(1-2x)}$$

$$c \quad x \mapsto \frac{x}{\sqrt{x^2 + 1}}$$

$$a \quad \text{Let } y = \sqrt{(1 + \sin^2 x)} = (1 + \sin^2 x)^{1/2}.$$

Using the chain rule we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2} \times \frac{d}{dx}(1 + \sin^2 x) \times (1 + \sin^2 x)^{-1/2} \\ &= \frac{1}{2} \times (2 \sin x \cos x) \times \frac{1}{\sqrt{(1 + \sin^2 x)}} \\ &= \frac{\sin x \cos x}{\sqrt{(1 + \sin^2 x)}} \end{aligned}$$

b Let $y = e^{x^3} \sin(1 - 2x)$.

Using the product and chain rules, we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(e^{x^3}) \times \sin(1 - 2x) + e^{x^3} \times \frac{d}{dx}(\sin(1 - 2x)) \\ &= 3x^2 e^{x^3} \sin(1 - 2x) + e^{x^3} \times -2 \cos(1 - 2x) \\ &= e^{x^3} (3x^2 \sin(1 - 2x) - 2 \cos(1 - 2x))\end{aligned}$$

c Let: (Quotient rule)

$$\begin{aligned}f(x) = \frac{x}{\sqrt{x^2 + 1}} &\Rightarrow f'(x) = \frac{\frac{d}{dx}(x) \times \sqrt{x^2 + 1} - x \times \frac{d}{dx}(\sqrt{x^2 + 1})}{(\sqrt{x^2 + 1})^2} \\ &= \frac{1 \times \sqrt{x^2 + 1} - x \times \frac{1}{2} \times 2x \times (x^2 + 1)^{-\frac{1}{2}}}{x^2 + 1} \\ &= \frac{\sqrt{x^2 + 1} - \frac{x^2}{\sqrt{x^2 + 1}}}{(x^2 + 1)} \\ &= \frac{(\sqrt{x^2 + 1})^2 - x^2}{(\sqrt{x^2 + 1})(x^2 + 1)} \\ &= \frac{1}{(\sqrt{x^2 + 1})(x^2 + 1)}\end{aligned}$$

Example E.2.8

Differentiate the following.

a $y = \ln\left(\frac{x}{x+1}\right), x > 0$ b $y = \sin(\ln t)$

c $y = x \ln(x^2)$

a $y = \ln\left(\frac{x}{x+1}\right) = \ln(x) - \ln(x+1)$
 $\therefore \frac{dy}{dx} = \frac{1}{x} - \frac{1}{x+1} = \frac{(x+1) - x}{x(x+1)} = \frac{1}{x(x+1)}$

Notice that using the log laws to first simplify this expression made the differentiation process much easier.

The other approach, i.e. letting $u = \frac{x}{x+1}$, $y = \ln(u)$ and

then using the chain rule would have meant more work – as not only would we need to use the chain rule but also the quotient rule to determine $\frac{du}{dx}$.

b Let $u = \ln t$ so that $y = \sin u$.

Using the chain rule we have:

$$\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt} = \cos(u) \times \frac{1}{t} = \frac{\cos(\ln t)}{t}$$

c Here we have a product $x \times \ln(x^2)$, so that the product rule needs to be used and then we need the chain rule to differentiate $\ln(x^2)$.

In this case we cannot simply rewrite $\ln(x^2)$ as $2\ln(x)$. Why?

Because the functions $\ln(x^2)$ and $2\ln(x)$ may have different domains. That is, the domain of $\ln(x^2)$ is all real values excluding zero (assuming an implied domain) whereas the domain of $2\ln(x)$ is only the positive real numbers. However, if it had been specified that $x > 0$, then we could have 'converted' $\ln(x^2)$ to $2\ln(x)$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x) \times \ln(x^2) + x \times \frac{d}{dx}(\ln(x^2)) = 1 \times \ln(x^2) + x \times \frac{2x}{x^2} \\ &= \ln(x^2) + 2\end{aligned}$$

Derivative of reciprocal circular functions

This is an application of the standard forms and the rules of differentiation. Whilst not specifically mentioned on SL 5.6, it can provide useful practice.

Dealing with the functions $\sec(x)$, $\cot(x)$ and $\operatorname{cosec}(x)$ is achieved by rewriting them as their reciprocals:

$$\sec(x) = \frac{1}{\cos(x)}, \cot(x) = \frac{1}{\tan(x)}, \operatorname{cosec}(x) = \frac{1}{\sin(x)}.$$

Once this is done, make use of the chain rule.

$$\begin{aligned}\text{Eg } \frac{d}{dx}(\operatorname{cosec} x) &= \frac{d}{dx}\left(\frac{1}{\sin x}\right) = \frac{d}{dx}[(\sin x)^{-1}] \\ &= -1 \times \cos x \times (\sin x)^{-2} = -\frac{\cos x}{(\sin x)^2}\end{aligned}$$

We could leave the answer as is or simplify it as follows:

$$\frac{\cos x}{\sin x \sin x} = -\cot x \operatorname{cosec} x.$$

Rather than providing a table of 'standard results' for the derivative of the reciprocal circular trigonometric functions, we consider them as special cases of the circular trigonometric functions.

Example E.2.9

Differentiate the following.

a $f(x) = \cot 2x, x > 0$

b $y = \sec^2 x$

c $y = \frac{\ln(\operatorname{cosec} x)}{x}$

$$\begin{aligned} \text{a } f(x) &= \cot 2x = \frac{1}{\tan 2x} = (\tan 2x)^{-1} \\ \therefore f'(x) &= -1 \times 2 \sec^2 2x \times (\tan 2x)^{-2} \\ &= -\frac{2 \sec^2 2x}{\tan^2 2x} \\ &= -2 \times \frac{1}{\cos^2 2x} \times \frac{1}{\tan^2 2x} \\ &= -2 \times \frac{1}{\cos^2 2x} \times \frac{\cos^2 2x}{\sin^2 2x} \\ &= -2 \operatorname{cosec}^2 2x \end{aligned}$$

$$\begin{aligned} \text{b } y &= \sec^2 x = \frac{1}{(\cos x)^2} = (\cos x)^{-2} \\ \therefore \frac{dy}{dx} &= -2 \times -\sin x \times (\cos x)^{-3} = \frac{2 \sin x}{(\cos x)^3} \\ &= 2 \times \frac{\sin x}{\cos x} \times \frac{1}{(\cos x)^2} \\ &= 2 \tan x \sec^2 x \end{aligned}$$

$$\begin{aligned} \text{c } y &= \frac{\ln(\operatorname{cosec} x)}{x} = \frac{\ln[(\sin x)^{-1}]}{x} = -\frac{\ln(\sin x)}{x} \\ \therefore \frac{dy}{dx} &= -\frac{\left(\frac{\cos x}{\sin x}\right) \times x - 1 \times \ln(\sin x)}{x^2} \\ &= \frac{x \cos x - \sin x \ln(\sin x)}{\sin x \cdot x^2} \\ &= \frac{x \cos x - \sin x \ln(\sin x)}{x^2 \sin x} \end{aligned}$$

An interesting result

A special case of the chain rule involves the case $y = x$.

By viewing this as an application of the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \text{ we have (after setting } y = x):$$

$$\frac{d(x)}{dx} = \frac{dx}{du} \cdot \frac{du}{dx} \Rightarrow 1 = \frac{dx}{du} \cdot \frac{du}{dx} \text{ i.e. } \frac{dx}{du} = 1 / \frac{du}{dx}$$

This important result is often written in the form: $\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$

We find that this result is useful with problems that deal with related rates.

Exercise E.2.3

1. Use the product rule to differentiate the following and then verify your answer by first expanding the brackets.

a $(x^2 + 1)(2x - x^3 + 1)$

b $(x^3 + x^2)(x^3 + x^2 - 1)$

c $\left(\frac{1}{x^2} - 1\right)\left(\frac{1}{x^2} + 1\right)$

d $(x^3 + x - 1)(x^3 + x + 1)$

2. Use the quotient rule to differentiate the following.

a $\frac{x+1}{x-1}$ b $\frac{x}{x+1}$ c $\frac{x+1}{x^2+1}$

d $\frac{x^2+1}{x^3-1}$ e $\frac{x^2}{2x+1}$ f $\frac{x}{1-2x}$

3. Differentiate the following.

a $e^x \sin x$ b $x \log_e x$

c $e^x(2x^3 + 4x)$ d $x^4 \cos x$

e $\sin x \cos x$ f $(1 + x^2) \tan x$

4. Differentiate the following.

a $\frac{x}{\sin x}$ b $\frac{\cos x}{x+1}$ c $\frac{e^x}{e^x + 1}$

d $\frac{\sin x}{\sqrt{x}}$ e $\frac{x}{\log_e x}$ f $\frac{\log_e x}{x+1}$

5. Differentiate the following.

a $e^{-5x} + x$

b $\sin 4x - \frac{1}{2} \cos 6x$

c $e^{\frac{1}{3}x} - \log_e(2x) + 9x^2$

- d $5 \sin(5x) + 3e^{2x}$
- e $\tan(4x) + e^{2x}$
6. Differentiate the following.
- a $\sin x^2 + \sin^2 x$ b $\tan(2\theta) + \frac{1}{\sin \theta}$
- c $\sin \sqrt{x}$ d $\cos\left(\frac{1}{x}\right)$
- e $\cos^3 \theta$ f $\sin(e^x)$
- g $\tan(\log_e x)$ h $\sqrt{\cos(2x)}$
7. Differentiate the following.
- a e^{2x+1} b $2e^{4-3x}$
- c $2e^{4-3x^2}$ d $\sqrt{e^x}$
- e $e^{\sqrt{x}}$ f $\frac{1}{2}e^{2x+4}$
- g $\frac{1}{2}e^{2x^2+4}$ h $\frac{2}{e^{3x+1}}$
- i e^{3x^2-6x+1} j $e^{\sin(\theta)}$
8. Differentiate the following.
- a $\log_e(x^2 + 1)$ b $\log_e(\sin \theta + \theta)$
- c $\log_e(e^x - e^{-x})$ d $\log_e\left(\frac{1}{x+1}\right)$
- e $(\log_e x)^3$ f $\sqrt{\log_e x}$
- g $\log_e(\sqrt{x-1})$ h $\log_e(1-x^3)$
9. Differentiate the following.
- a $x \log_e(x^3 + 2)$ b $\sqrt{x} \sin^2 x$
- c $\cos^2 \sqrt{\theta}$ d $x^3 e^{-2x^2+3}$
- e $\cos(x \log_e x)$ f $\log_e(\log_e x)$
- g $\frac{x^2 - 4x}{\sin(x^2)}$ h $\frac{10x + 1}{\log_e(10x + 1)}$
10. Find the value of x where the function $x \mapsto x e^{-x}$ has a horizontal tangent.
11. Find the gradient of the function $x \mapsto \sin\left(\frac{1}{x}\right)$, where $x = \frac{2}{\pi}$.
12. Find the gradient of the function $x \mapsto \log_e(x^2 + 4)$ at the point where the function crosses the y -axis.
13. For what value(s) of x will the function $x \mapsto \ln(x^2 + 1)$ have a gradient of 1.
14. Find the rate of change of the function $x \mapsto e^{-x^2} + 2$ at the point $(1, e)$.
15. Find: a $\frac{d}{dx}(\sin x \cos x)$ b $\frac{d}{dx}(\sin x^\circ)$
16. a
- If y is the product of three functions, i.e. $y = f(x)g(x)h(x)$, show that:
- $$\frac{dy}{dx} = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).$$
- b Hence, differentiate the following:
- i $x^2 \sin x \cos x$ ii $e^{-x^3} \sin(2x) \log_e(\cos x)$
17. a Given that $f(x) = 1 - x^3$ and $g(x) = \log_e x$, find:
- i $(f \circ g)'(x)$ ii $(g \circ f)'(x)$
- b Given that $f(x) = \sin(x^2)$ and $g(x) = e^{-x}$, find:
- i $(f \circ g)'(x)$ ii $(g \circ f)'(x)$
18. Given that $T(\theta) = \frac{\cos k\theta}{2 + 3 \sin k\theta}$, $k \neq 0$, determine $T'\left(\frac{\pi}{2k}\right)$.
19. If $f(x) = (x-a)^m(x-b)^n$, find x such that $f'(x) = 0$
20. If $f(\theta) = \sin \theta^m \cos \theta^n$, find θ such that $f'(\theta) = 0$.

21. Differentiate the following.

a $f(x) = \cot 4x$ b $g(x) = \sec 2x$

c $f(x) = \operatorname{cosec} 3x$ d $y = \sin\left(3x + \frac{\pi}{2}\right)$

22. Differentiate the following.

a $\sec x^2$ b $\sin x \sec x$

c $\ln(\sec x)$ d $\cot^3 x$

e $\frac{x}{\operatorname{cosec} x}$ f $\frac{\operatorname{cosec} x}{\sin x}$

Extra questions



Higher Derivatives

Since the derivative of a function f is another function, f' , then it may well be that this derived function can itself be differentiated. If this is done, we obtain the second derivative of f which is denoted by f'' and read as “ f -double-dash”.

The following notation for $y = f(x)$ is used:

First derivative $\frac{dy}{dx} = f'(x) [= y']$

Second derivative $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x) [= y'']$

So, for example, if $f(x) = x^3 - 5x^2 + 10$

then $f'(x) = 3x^2 - 10x$ and $f''(x) = 6x - 10$.

The expression $\frac{d^2y}{dx^2}$ is read as “dee-two- y by dee- x -squared” and the expression y'' is read as “ y -double-dash”.

Example E.2.10

Find the second derivative of:

a $x^4 - \sin 2x$

b $\ln(x^2 + 1)$

c $x \times e^{2x}$

a Let $y = x^4 \sin 2x$ then $y' = 4x^3 - 2 \cos 2x$ and $y'' = 12x^2 + 4 \sin 2x$.

b Let $f(x) = \ln(x^2 + 1)$ then $f'(x) = \frac{2x}{x^2 + 1}$ and $f''(x) = \frac{2(x^2 + 1) - 2x(2x)}{(x^2 + 1)^2} = \frac{2 - 2x^2}{(x^2 + 1)^2}$

c $f(x) = x \times e^{2x}$
 $f'(x) = 1 \times e^{2x} + x \times 2e^{2x}$
 $= e^{2x} + 2xe^{2x}$
 $= e^{2x}(1 + 2x)$
 $f''(x) = 2e^{2x}(1 + 2x) + e^{2x} \times 2$
 $= e^{2x}(2 + 4x + 2)$
 $= e^{2x}(4x + 4)$

As we can see from the above examples, some second derivatives require the use of algebra to obtain a simplified answer.

Note then that, just as we can find the second derivative, so too can we determine the third derivative and the fourth derivative and so on (of course, assuming that these derivatives exist). We keep differentiating the results. The notation then is extended as follows:

Third derivative is $f'''(x)$ (“ f -triple-dash”) and so on where the n th derivative is $f^{(n)}(x)$ or $\frac{d^n y}{dx^n}$.

Exercise E.2.4

1. Find the second derivative of the following functions.

a $f(x) = x^5$

b $y = (1 + 2x)^4$

c $f: x \mapsto \frac{1}{x}$ where $x \in \mathbb{R}$

d $f(x) = \frac{1}{1 + x}$

e $y = (x - 7)(x + 1)$

f $f: x \mapsto \frac{x + 1}{x - 2}$ where $x \in \mathbb{R} \setminus \{2\}$

g $f(x) = \frac{1}{x^6}$

h $y = (1 - 2x)^3$

i $y = \ln x$

j $f(x) = \ln(1-x^2)$

k $y = \sin 4\theta$

l $f(x) = x \sin x$

2. Find the second derivative (with respect to x) of the following.

a $x^3 \sin x$

b $x^2 \ln x$

c $\frac{x+1}{x+3}$

d $e^{\sin x}$

e $\sqrt{x+1}$

f $\frac{e^{2x}}{x+1}$

3. Find the second derivative of the function $f(x) = \frac{\log_e x}{x^2}$

Find a formula for the second derivative of the function $f(x) = \frac{\log_e x}{x^n}$.

4. Consider the function $f(x) = \frac{1}{x+1}, x \neq -1$.

Find the first five derivatives by differentiating the function five times. Hypothesise a formula for the n th derivative of this function. Use the method of mathematical induction or other appropriate method to prove that your formula works for all whole numbered values of n .

5. Find a formula for the second derivative of the family of functions $f(x) = \left(\frac{x+1}{x-1}\right)^n$ where n is a real number.

6. Given $y = \frac{1}{1-x}$, prove that $\frac{d^n y}{dx^n} = \frac{n!}{(1-x)^{n+1}}$ for $n \geq 1$.

Extra questions



Graphical Interpretation

The properties of the basic function are the first things to look at when sketching curves. The main features are summarised below:

- $y = f(x)$ is the equation of the function
- $f(0)$ - intercept(s) for $f(x)$
- $f(x) = 0$ - intercept(s) for $f(x)$
- determine the domain and range for $f(x)$
- identify the asymptotes for $f(x) = \frac{p(x)}{q(x)}$
- vertical asymptote(s): the solution(s) for $q(x) = 0$
- horizontal asymptote: if $\deg_{p(x)} = \deg_{q(x)}$ then dividing out the leading (highest power) coefficients between $p(x)$ and $q(x)$, gives the asymptote:

$$y = \frac{\text{leading coeff. } p(x)}{\text{leading coeff. } q(x)}$$
- also, if $\deg_{p(x)} < \deg_{q(x)}$, then the asymptote is:

$$y = \lim_{x \rightarrow \pm\infty} f(x)$$
- oblique asymptote: if $\deg_{p(x)} > \deg_{q(x)}$, the quotient $\frac{p(x)}{q(x)}$ is the asymptote

First Derivative

The properties of the first derivative also help us sketch curves. The main points are:

- $f'(x)$ suggests the monotonicity (either increasing or decreasing) of $f(x)$ in the interval $S[a, b]$.
 - $f'(x) > 0$: $f(x)$ is increasing in S .
 - $f'(x) < 0$: $f(x)$ is decreasing in S .
- $f'(x)$ detects the existence of inflection point(s) in the interval $S[a, b]$.
 - $f'(k) = 0$ but sign of $f'(x)$ does not change for $x > k$ and $x < k$
- First Derivative Test for Optimum for $k \in S$
- $f(k)$ maximum: $f'(x) > 0$ for $x < k$

$$f'(x) < 0 \text{ for } x > k$$

$$f'(k) = 0$$

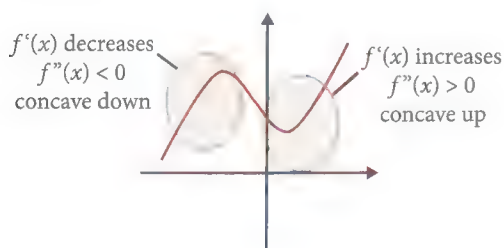
- $f(k)$ minimum: $f'(x) < 0$ for $x < k$

$$f'(x) > 0 \text{ for } x > k$$

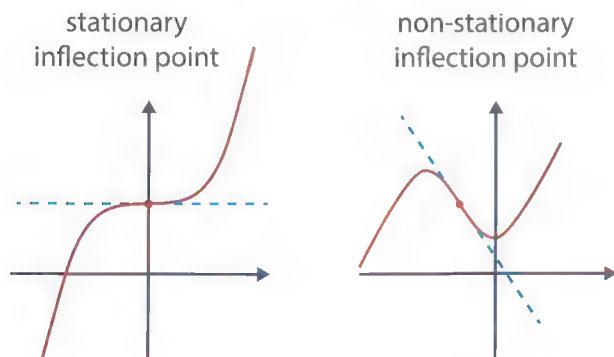
$$f'(k) = 0$$

Second Derivative

- $f''(x)$ suggests the concavity of $f(x)$ in the interval $S[a, b]$.
- $f''(x) > 0$: $f(x)$ concaves upward in S .
- $f''(x) < 0$: $f(x)$ concaves downward in S .



- $f''(x)$ determines the nature of an inflection point in the interval in the interval $S[a, b]$.
- $f''(k)$ changes sign, $f''(k) = 0$ and $f'(k) = 0$, then $f(k)$ is a stationary inflection point
- $f''(k)$ changes sign, $f''(k) = 0$ and $f'(k) \neq 0$, then $f(k)$ is a non-stationary inflection point



- Second Derivative Test for Optimum for $k \in S$

- $f(k)_{\max}$ $f''(k) < 0$ and $f'(k) = 0$

- $f(k)_{\min}$ $f''(k) > 0$ and $f'(k) = 0$

Other Factors

- global minimum is the minimum value of $y = f(x)$ on the entire domain.
- local minimum is the a turning point $y = f(x)$ at $x = k$, for $f'(x)$ changes from negative to positive.
- global maximum is the maximum value of $y = f(x)$ on the entire domain.
- local maximum is the a turning point $y = f(x)$ at $x = k$, for $f'(x)$ changes from positive to negative.

The following table summarises the effects of the first and second derivative on the curvature of a continuous function

	$f'(x) > 0$	$f'(x) < 0$
$f''(x) > 0$		
$f''(x) < 0$		

Exercise E.2.5

- Find the overall maximum of these functions:

- $f(x) = x^3 - x - 7, x \in [1, 6]$

- $f(x) = 2 + x - x^2, x \in [-2, 5]$

- $f(x) = \frac{x^2 - 1}{2x + 1}, x \in [2, 6]$

- $f(x) = x^2 \sin(2x), x \in [0, 6]$

- $f(x) = \frac{e^x}{\cos x}, x \in [1, 1.4]$

- $f(x) = \sin(e^x), x \in [1, 6]$



What can be inferred if we only have information about the rate at which events occur.

Here are a couple of situations in which rate of change information is collected and used to infer important conclusions.

Inertial Navigation

Spacecraft



and commercial aircraft



need accurate methods of navigation.

There are two categories of navigation devices: internal and external.

An **internal navigation system** is contained entirely within the vehicle. Dead reckoning navigation depends on compass (direction), speed (ie. a rate) and time to calculate position. Taking 'sights' of the stars also allows position to be calculated.

External navigation systems depend on signals generated outside the vehicle. Global Positioning System (GPS) is the method we are most familiar with. It appears to work entirely inside the device, but this is an illusion as it depends on signals received from specialised satellites. If these are shielded (eg. in caves and tunnels) the system fails. Older systems such as NDB and VOR use land based signals.

The advantage of internal systems is that they do not fail as a result of a breakdown outside the vehicle or shielding by water, rock, electrical storms etc.

The most sophisticated such system, called **Inertial Navigation**, depends on measuring acceleration (a second derivative of position), using it to infer speed (first derivative) and hence position. The heart of such a device is a very sensitive accelerometer (meter for measuring acceleration).

Chemical Catalysis

A catalyst is a substance that alters the rate of a chemical reaction without actually being consumed in the process.



Motor vehicle exhaust pipes often contain a catalyst that helps reduce the pollution resulting from the exhaust gases. Large scale chemical manufacture, as in oil refineries, depends heavily on catalysis. What factors are involved in making decisions about which catalysts to use?

Decompression Sickness when SCUBA diving.

Dive graph video.

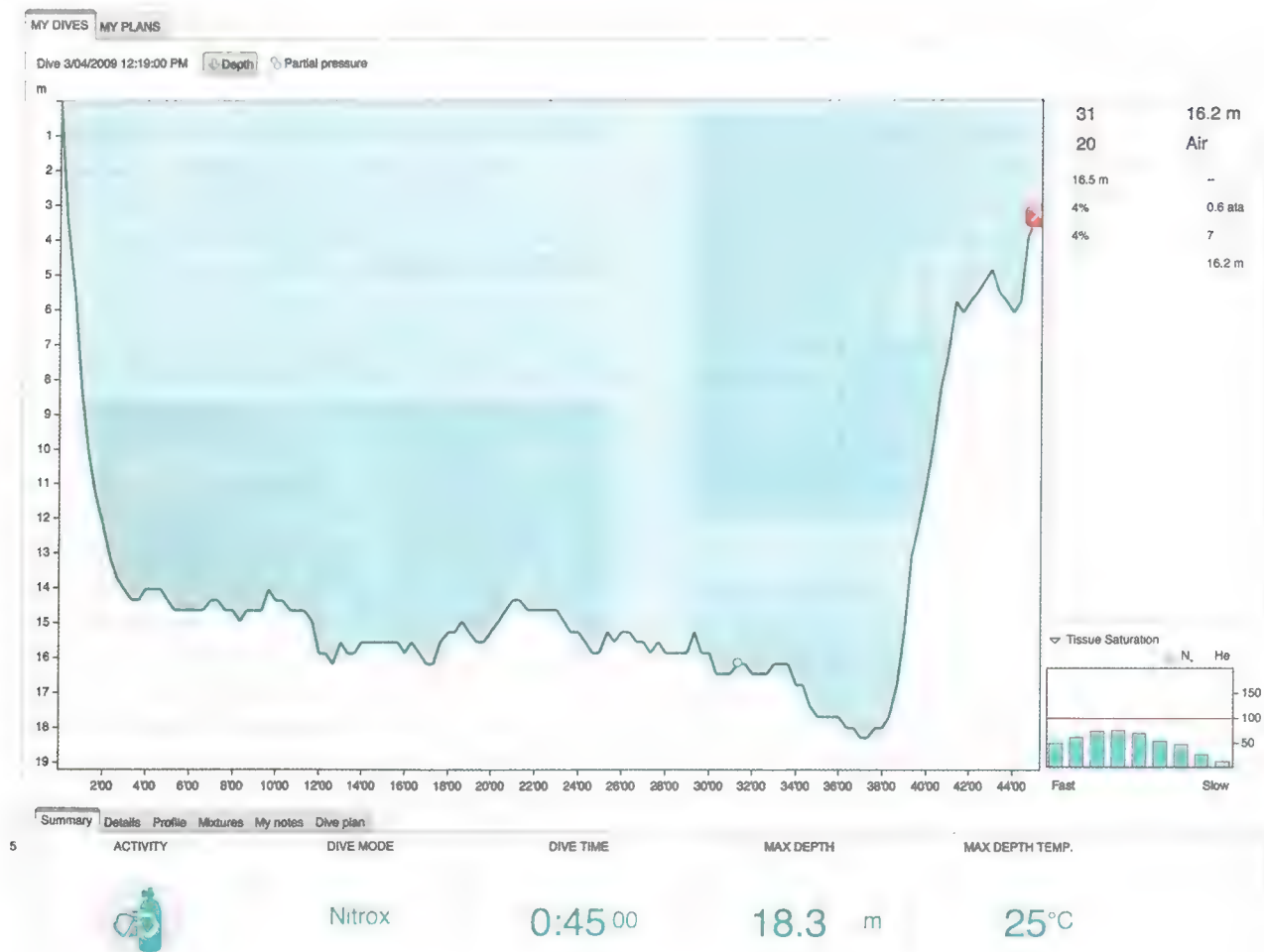


This is an issue of rates.

- The rate at which nitrogen dissolves in a diver's tissues due to the pressure at depth.
- The rate at which nitrogen is released as the diver ascends.
- The rate at which the diver ascends.

The graph shows what is known as a dive profile. Please watch the video and use pause to get an idea of the rates of change involved. You are looking at both the depth graph and the small bar chart near the bottom right. This gives you the tissue saturation.

Answers



E.4 Optimization

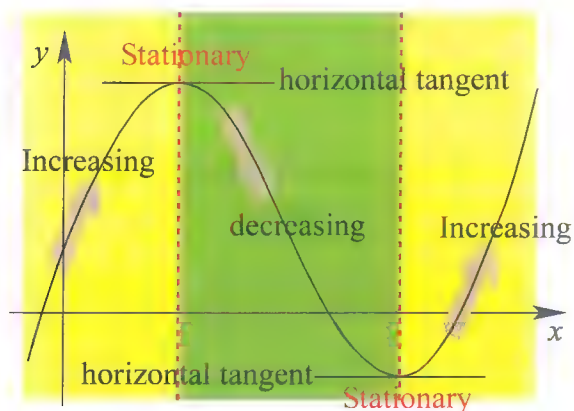
SL 5.8

Stationary Points

In the Common Core Text, we discussed the conditions for a function to be increasing ($f'(x) > 0$) and for a function to be decreasing ($f'(x) < 0$). What happens at the point where a function changes from an increasing state ($f'(x) > 0$) to $f'(x) = 0$ and then to a decreasing state ($f'(x) < 0$) or vice versa?

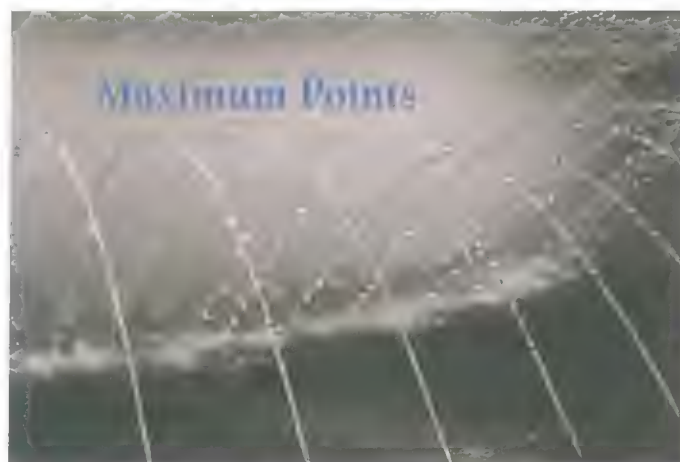
Points where this happens are known as stationary points. At the point where the function is in a state where it is neither increasing nor decreasing, we have that $f'(x) = 0$. There are times when we can call these stationary points **stationary points**, but on such occasions, we prefer the terms local maximum and local minimum points.

At the point(s) where $\frac{dy}{dx} = f'(x) = 0$ we have a stationary point.



There are three types of stationary points,

namely; local maximum point,
 local minimum point and
 stationary point of inflection.



1. Local maximum

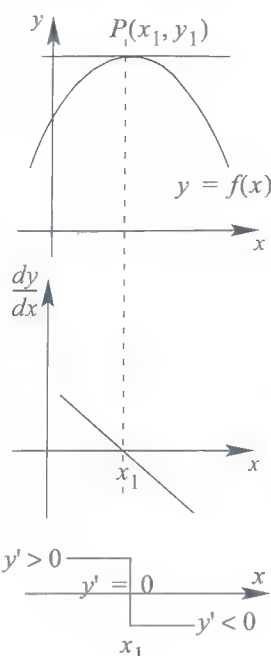
When sketching a curve, if the following properties hold:

- i. At $P(x_1, y_1)$,
 $\frac{dy}{dx} = f'(x) = 0$ that is $f'(x_1) = 0$.
- ii. For $x < x_1$ then $\frac{dy}{dx} > 0$
 $x > x_1$ then $\frac{dy}{dx} < 0$

Where the two chosen values of x are such that one is just slightly less than x_1 and the other is just slightly greater than x_1 , then, $y = f(x)$ has a local maximum point (also known as a relative maximum) at the point $P(x_1, y_1)$.

iii. Graph of the gradient function:

Notice that the values of $\frac{dy}{dx}$ are changing from positive to negative. Sometimes this is referred to as the sign of the first derivative. At this stage, it isn't so much the magnitude of the derivative that is important, but that there is a change in the sign of the derivative near $x = x_1$.



In this instance the sign of the derivative changes from positive to negative.

This change in sign is sometimes represented via the above diagram, which is referred to as a sign diagram of the first derivative. Such diagrams are used to confirm the nature of stationary points (in this case, that a local maximum occurs at $x = x_1$).

Example E.4.1

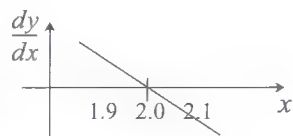
Find the local maximum value of the function whose equation is $f(x) = -3 + 4x - x^2$.

First we differentiate: $f(x) = -3 + 4x - x^2 \Rightarrow f'(x) = 4 - 2x$

Next, equate $f'(x)$ to 0 and solve for x :

$$0 = 4 - 2x \Leftrightarrow x = 2$$

To ensure that we have obtained a local maximum we choose values of x slightly less than 2 and slightly greater than 2, for example, choose $x = 1.9$ and $x = 2.1$.



For $x = 1.9$, we have that $f'(1.9) = 4 - 2(1.9) = 0.2$.

For $x = 2.1$, we have that $f'(2.1) = 4 - 2(2.1) = -0.2$.

Using the graph of the gradient function, $\frac{dy}{dx}$, confirms that there is a local maximum at $x = 2.0$.

The local maximum value of $f(x)$, is found by substituting $x = 2$ into the given equation: $f(2) = -3 + 4(2) - (2)^2 = 1$. That is, the local maximum occurs at the point $(2, 1)$.

2. Local minimum

When sketching a curve, if the following properties hold:

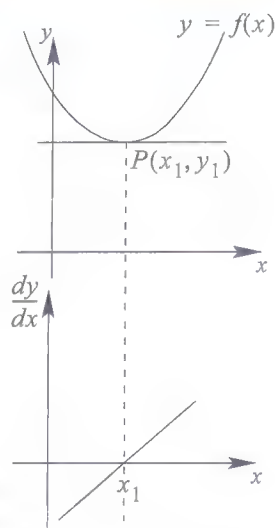
i. At $P(x_1, y_1)$,

$$\frac{dy}{dx} = f'(x) = 0 \text{ that is } f'(x_1) = 0.$$

ii. For $x > x_1$ then $\frac{dy}{dx} > 0$

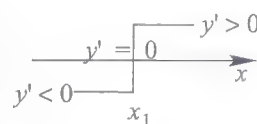
$$x < x_1 \text{ then } \frac{dy}{dx} < 0$$

where the two chosen values of x are such that one is just slightly greater than x_1 and the other is just slightly less than x_1 , then $y = f(x)$ has a local minimum point (also known as a relative minimum) at the point $P(x_1, y_1)$.



iii. Graph of the gradient function:

Notice that the values of $\frac{dy}{dx}$ are changing from negative to positive. Sometimes this is referred to as the sign of the first derivative.



Again we can represent the change in the sign of the first derivative via the diagram alongside, which is referred to as a sign diagram of the first derivative. Such diagrams are used to confirm the nature of stationary points (in this case, that a local minimum occurs at $x = x_1$).

3. Points of inflection

There are two types:

- A. Stationary points of inflection
- B. Non-stationary points of inflection

A. Stationary point of inflection

The following properties hold at a stationary point of inflection.

- i At $P(x_1, y_1)$, $f'(x) = 0$. That is $f'(x_1) = 0$.
- ii For $x < x_1$, $f'(x) > 0$ and for $x > x_1$, $f'(x) > 0$.

Similarly,

At $P(x_2, y_2)$, $f'(x) = 0$. That is $f'(x_2) = 0$

and for $x < x_2$, $f'(x) < 0$ and for $x > x_2$, $f'(x) < 0$.

That is, the gradient of the curve on either side of x_1 (or x_2) has the same sign.

- iii Graph of the gradient function, $y = f'(x)$:

Notice that the values of $f'(x)$ have the same sign on either side of $x = x_1$.

Notice that at $x = x_1$, the gradient of $f'(x)$ is also equal to zero. That is, the derivative of the derivative is equal to zero.

Therefore if there is a stationary point of inflection at $x = x_1$ then $f''(x_1) = 0$.

First we differentiate (using the product rule):

$$\begin{aligned} y &= (x-1)^3(x+2) \Rightarrow \frac{dy}{dx} = 3(x-1)^2(x+2) + (x-1)^3(1) \\ &= (x-1)^2[3(x+2) + (x-1)] \\ &= (x-1)^2(4x+5) \end{aligned}$$

Solving for $\frac{dy}{dx} = 0$, we have,

$$(x-1)^2(4x+5) = 0 \Leftrightarrow x = 1 \text{ or } x = -\frac{5}{4} (= -1.25).$$

We can now check the sign of the derivative on either side of $x = 1$ and $x = -1.25$

At $x = 1$:

$$\text{For } x = 0.9, \frac{dy}{dx} = (-0.1)^2(8.6) = 0.086 > 0$$

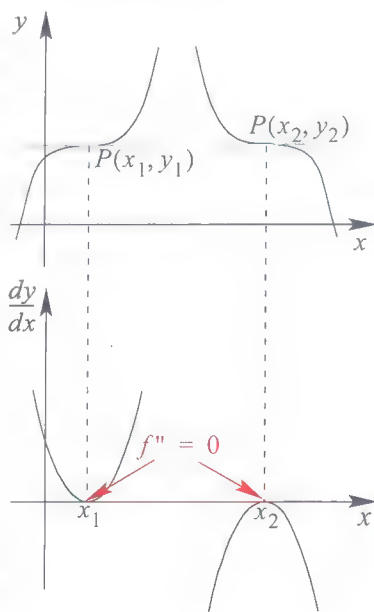
$$\text{For } x = 1.1, \frac{dy}{dx} = (0.1)^2(9.4) = 0.094 > 0.$$

As the sign of the first derivative is the same on either side of $x = 1$, we have a stationary point of inflection at $x = 1$, i.e. at $(1, 0)$.

A sketch of the graph of $y = (x-1)^3(x+2)$ quickly confirms our result.

For $x = -1.25$, the graph shows a local minimum occurring at this point.

Notice that the sign diagrams of the first derivative in Examples E.4.1 and E.4.2 look slightly different. We did this because, as long as the diagram provides a clear indication of the sign of the first derivative, then its appearance can vary.



Example E.4.2

Find the stationary point of inflection for the graph with equation $y = (x-1)^3(x+2)$.

B. Non-stationary point of inflection

The following properties hold at a non-stationary point of inflection:

- i. At $P(x_1, y_1)$, $f'(x_1) \neq 0$ and $f''(x_1) = 0$.
- ii. For $x < x_1$, $f'(x) > 0$ and for $x > x_1$, $f'(x) > 0$.

Similarly,

At $P(x_2, y_2)$, $f'(x_2) \neq 0$ and $f''(x_2) = 0$.

For $x < x_2$, $f'(x) < 0$ and for $x > x_2$, $f'(x) < 0$.

That is, the gradients of the curve on either side of x_1 (or x_2) have the same sign.

- iii. Graph of the gradient function, $y = f'(x)$:

Notice that the values of $f'(x)$ have the same sign on either side of $x = x_1$.

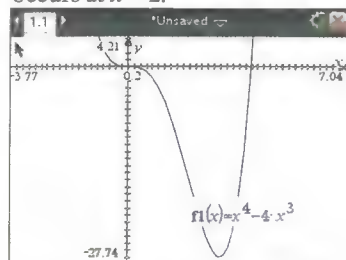
Example E.4.3

Show that the curve with equation $y = x^4 - 4x^3$ has a non-stationary point of inflection at $x = 2$.

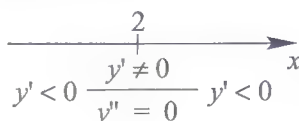
For the curve to have a non-stationary point of inflection at $x = 2$ we need to show that:

1. $\frac{d^2y}{dx^2} = 0$ at $x = 2$ and
2. the sign of the gradient, $\frac{dy}{dx}$, is the same on both sides of $x = 2$ and
3. that $\frac{dy}{dx} \neq 0$ when $x = 2$.

A quick sketch of the function indicates that a point of inflection occurs at $x = 2$:



Sign diagram:



$$\begin{aligned}\frac{dy}{dx} &= 4x^3 - 12x^2 \\ \Rightarrow \frac{d^2y}{dx^2} &= 12x^2 - 24x\end{aligned}$$

$$\text{For } x = 2, \frac{d^2y}{dx^2} = 12(2)^2 - 24(2) = 0 \text{ and } \frac{dy}{dx} = -16 \neq 0.$$

$$\text{For } x = 2.1, \frac{dy}{dx} = -15.876 \text{ and for } x = 1.9, \frac{dy}{dx} = -15.88.$$

Therefore there is a non-stationary point of inflection at $x = 2$.

It is important to realize that it is not sufficient to say that "If $f''(x) = 0$ at $x = a$ then there must be a point of inflection at $x = a$."

Does $f''(a) = 0$ imply there is a point of inflection at $x = a$?

The answer is NO!

Although it is necessary for the second derivative to be zero at a point of inflection, the fact that the second derivative is zero at $x = a$ does not mean there must be a point of inflection at $x = a$.

That is:

$f''(a) = 0$ is a necessary but not a sufficient reason for there to be an inflection point at $x = a$.

We use the following example to illustrate this.

Consider the case where $f(x) = x^4$.

Now, $f''(x) = 12x^2$, therefore solving for $f''(x) = 0$,

we have $12x^2 = 0 \Leftrightarrow x = 0$.

That is, $f''(0) = 0$. So, do we have a point of inflection at $x = 0$?

A sketch of f shows that although $f''(x) = 0$ at $x = 0$, there is in fact a local minimum and not a point of inflection at $x = 0$.

In other words, finding where $f''(x) = 0$ is not enough to indicate that there is an inflection point. To determine if there is a point of inflection you need to check the sign of the first derivative on either side of the x -value in question.

Example E.4.4

Find and classify all stationary points (including inflection points) of $f(x) = x^3 - 3x^2 - 9x + 1$.

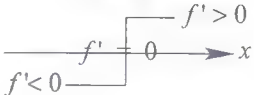
Now, $f(x) = x^3 - 3x^2 - 9x + 1 \Rightarrow f'(x) = 3x^2 - 6x - 9$

Solving for stationary points we have,

$$3x^2 - 6x - 9 = 0 \Leftrightarrow 3(x-3)(x+1) = 0$$

$$\Leftrightarrow x = 3 \text{ or } x = -1$$

So that $f(3) = -26$ and $f(-1) = 6$.



Using the sign of the first derivative, we have:

At $x = 3$:

For $x < 3$ ($x = 2.9$) $f'(2.9) < 0$ and for $x > 3$ ($x = 3.1$) $f'(3.1) > 0$

Therefore, there is a local minimum at $(3, -26)$.

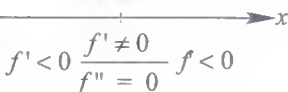
At $x = -1$:

For $x < -1$ ($x = -1.1$) $f'(-1.1) > 0$ and for $x > -1$ ($x = -0.9$) $f'(-0.9) < 0$.

Therefore, there is a local maximum at $(-1, 6)$.

A maximum may be referred to as an 'optimum'.

Checking for inflection points:



We have: $f''(x) = 0 \Leftrightarrow 6x - 6 = 0 \Leftrightarrow x = 1$.

For $x < 1$ ($x = 0.9$) $f''(0.9) < 0$ and

for $x > 1$ ($x = 1.1$) $f''(1.1) > 0$.

As the sign of the first derivative remains the same on either side of $x = 1$, there is a point of inflection at $(1, -10)$. Then, as the **first derivative** at $x = 1$ is **not zero**, we have a **non-stationary** point of inflection at $x = 1$.

Curve Sketching

The properties of the basic function are the first things to look at when sketching curves. The main features are summarised below:

- $y = f(x)$ is the equation of the function
- $f(0)$ - intercept(s) for $f(x)$
- $f(x) = 0$ - intercept(s) for $f(x)$
- determine the domain and range for $f(x)$
- identify the asymptotes for $f(x) = \frac{p(x)}{q(x)}$
- vertical asymptote(s): the solution(s) for $q(x) = 0$
- horizontal asymptote: if $\deg_{p(x)} = \deg_{q(x)}$ then dividing out the leading (highest power) coefficients between $p(x)$ and $q(x)$, gives the asymptote:

$$y = \frac{\text{leading coeff. } p(x)}{\text{leading coeff. } q(x)}$$

- also, if $\deg_{p(x)} < \deg_{q(x)}$, then the asymptote is:

$$y = \lim_{x \rightarrow \pm\infty} f(x)$$

- oblique asymptote: if $\deg_{p(x)} > \deg_{q(x)}$, the quotient $\frac{p(x)}{q(x)}$ is the asymptote

First Derivative

The properties of the first derivative also help us sketch curves. The main points are:

- $f'(x)$ suggests the monotonicity (either increasing or decreasing) of $f(x)$ in the interval $S[a, b]$.
 - $f'(x) > 0$: $f(x)$ is increasing in S .
 - $f'(x) < 0$: $f(x)$ is decreasing in S .
- $f'(x)$ detects the existence of inflection point(s) in the interval $S[a, b]$.
 - $f'(k) = 0$ but sign of $f'(x)$ does not change for $x > k$ and $x < k$

- First Derivative Test for Optimum for $k \in S$

$f(k)$ maximum: $f'(x) > 0$ for $x < k$

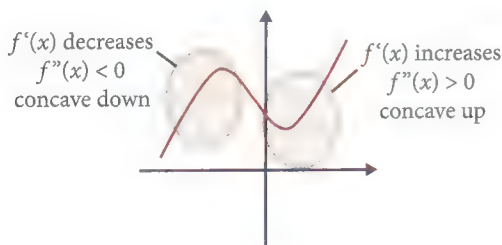
$$f'(x) < 0 \text{ for } x > k$$

$$f'(k) = 0$$

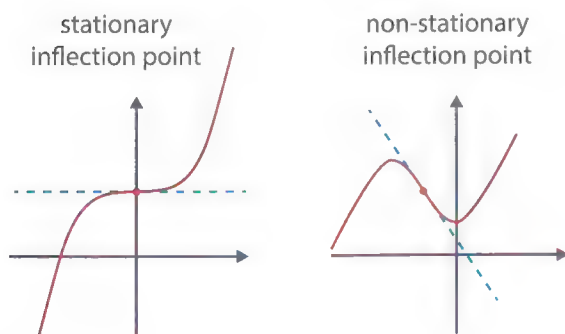
- $f(k)$ minimum: $f'(x) < 0$ for $x < k$
 $f'(x) > 0$ for $x > k$
 $f'(k) = 0$

Second Derivative (Extension Material)

- $f''(x)$ suggests the concavity of $f(x)$ in the interval $S[a, b]$.
- $f''(x) > 0$: $f(x)$ concaves upward in S .
- $f''(x) < 0$: $f(x)$ concaves downward in S .



- $f''(x)$ determines the nature of an inflection point in the interval in the interval $S[a, b]$.
- $f''(k)$ changes sign, $f''(k) = 0$ and $f'(k) = 0$, then $f(k)$ is a stationary inflection point
- $f''(k)$ changes sign, $f''(k) = 0$ and $f'(k) \neq 0$, then $f(k)$ is a non-stationary inflection point



- Second Derivative Test for Optimum for $k \in S$

- $f(k)_{\max}$ $f''(k) < 0$ and $f'(k) = 0$

- $f(k)_{\min}$ $f''(k) > 0$ and $f'(k) = 0$

•

•

Other Factors

- global minimum is the minimum value of $y = f(x)$ on the entire domain.
- local minimum is the a turning point $y = f(x)$ at $x = k$, for $f'(x)$ changes from negative to positive.
- global maximum is the maximum value of $y = f(x)$ on the entire domain.
- local maximum is the a turning point $y = f(x)$ at $x = k$, for $f'(x)$ changes from positive to negative.

The following table summarises the effects of the first and second derivative on the curvature of a continuous function

	$f'(x) > 0$	$f'(x) < 0$
$f''(x) > 0$		
$f''(x) < 0$		

Example E.4.5

Find the minimum value of $x \mapsto \frac{e^x}{x}$, $x > 0$

First differentiate (using the quotient rule):

$$\frac{d}{dx}\left(\frac{e^x}{x}\right) = \frac{\frac{d}{dx}(e^x) \times x - e^x \times \frac{d}{dx}(x)}{x^2} = \frac{e^x x - e^x}{x^2} = \frac{e^x(x-1)}{x^2}$$

We solve for $\frac{d}{dx}\left(\frac{e^x}{x}\right) = 0$, i.e. $\frac{e^x(x-1)}{x^2} = 0 \Leftrightarrow e^x(x-1) = 0$

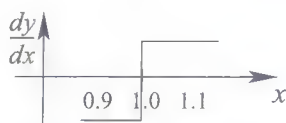
However, $e^x \neq 0$ for all real values of x , therefore, the only possible solution occurs if $x = 1$.

To verify that we have a local minimum we select a value of x slightly less than $x = 1$ and one slightly greater than $x = 1$:

For ($x < 1$): choose $x = 0.9$, we have that $\frac{d}{dx}\left(\frac{e^x}{x}\right) = -0.30$.

For ($x > 1$): choose $x = 1.1$, we have that $\frac{d}{dx}\left(\frac{e^x}{x}\right) = 0.25$.

Therefore, for $x = 1$ we have a local minimum point.



The minimum value is therefore given by:

Sign diagram of first derivative:

$y = \frac{e^1}{1} = e$, and occurs at the point $(1, e)$.

Example E.4.6

Find the local minimum of the function:

$$y = \sin(x) + \frac{1}{2}\sin(2x), 0 \leq x \leq 2\pi.$$

We start by differentiating and finding the stationary points (i.e. solving for $\frac{dy}{dx} = 0$):

$$\text{Now, } y = \sin(x) + \frac{1}{2}\sin(2x) \Rightarrow \frac{dy}{dx} = \cos(x) + \cos(2x).$$

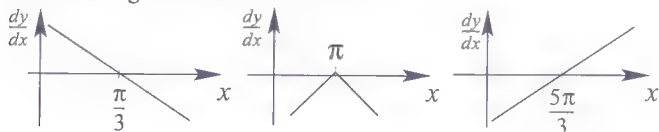
Therefore, solving we have $\cos(x) + \cos(2x) = 0$

$$\begin{aligned}\cos(x) + (2\cos^2(x) - 1) &= 0 \\ (2\cos(x) - 1)(\cos(x) + 1) &= 0\end{aligned}$$

Therefore, $\cos(x) = \frac{1}{2}$ or $\cos(x) = -1$, i.e. $x = \frac{\pi}{3}, \frac{5\pi}{3}$ or

$x = \pi$ for $x \in [0, 2\pi]$.

We can check the nature of the stationary points by making use of the sign of the first derivative:



The graph of the gradient function (near $x = \frac{5\pi}{3}$) indicates that a local minimum occurs at $x = \frac{5\pi}{3}$. And so, the local minimum value is given by:

$$y = \sin\left(\frac{5\pi}{3}\right) + \frac{1}{2}\sin\left(\frac{10\pi}{3}\right) = -\frac{3\sqrt{3}}{4}.$$

NB: In the process we have come across a new sign diagram (at $x = \pi$). This is dealt with in the next section.

Example E.4.7

Locate the stationary points of inflection for the curve $f(x) = x^3e^{-x}$.

We begin by determining where stationary points occur:

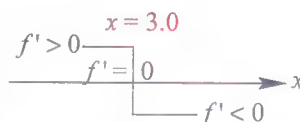
$$f(x) = x^3e^{-x} \Rightarrow f'(x) = 3x^2e^{-x} - x^3e^{-x}$$

Setting $f'(x) = 0$, we have:

$$3x^2e^{-x} - x^3e^{-x} = 0 \Leftrightarrow x^2e^{-x}(3 - x) = 0$$

$$\Leftrightarrow x = 0 \text{ or } x = 3$$

We can use the sign of the first derivative to help us determine the nature of the stationary point.



At $x = 3$: Sign diagram:

$$\text{For } x = 2.9, f'(2.9) = (2.9)^2e^{-2.9}(3 - 2.9) = 0.046 > 0$$

$$\text{For } x = 3.1, f'(3.1) = (3.1)^2e^{-3.1}(3 - 3.1) = -0.043 < 0$$

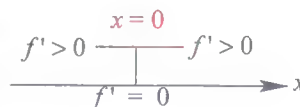
Therefore there exists a local maximum at $x = (3, 1)$.

At $x = 0$:

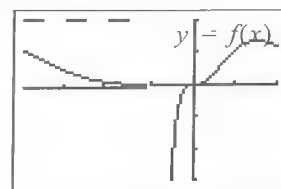
$$\text{For } x = 0.1, f'(0.1) = (0.1)^2e^{-0.1}(3 - 0.1) = 0.026 > 0$$

$$\text{For } x = -0.1, f'(-0.1) = (-0.1)^2e^{0.1}(3 + 0.1) = 0.034 > 0$$

As there is no change in the sign of the first derivative there is a stationary point of inflection at $x = 0$.



Alternatively, we could sketch a graph of the function and use it to help us determine where the stationary point of inflection occurs.



From the graph we can see that there is a local maximum at $x =$

3 and a stationary point of inflection at $x = 0$.

Therefore, the stationary point of inflection occurs at $(0, 0)$.

Exercise E.4.1

1. Sketch of the graph of the function $f(x), x \in \mathbb{R}$, where:

a $f(1) = 2, f'(1) = 0, f(3) = -2, f'(3) = 0$,
 $f'(x) < 0$ for $1 < x < 3$ and $f'(x) > 0$ for $x > 3$ and $x < 1$.

b $f'(2) = 0, f(2) = 0, f'(x) > 0$ for $0 < x < 2$
and $x > 2, f'(x) < 0$ for $x < 0$ and $f(0) = -4$.

c $f(4) = f(0) = 0, f'(0) = f'(3) = 0$,
 $f'(x) > 0$ for $x > 3$ and $f'(x) < 0$ for $x < 0$ and $0 < x < 3$.

d $f(4) = 4, f'(x) > 0$ for $x > 4, f'(x) < 0$
for $x < 4$, as $x \rightarrow 4^+, f'(x) \rightarrow +\infty$ and as
 $x \rightarrow 4^-, f'(x) \rightarrow -\infty$.

2. Find the coordinates and nature of the stationary points for the following:

a $y = 3 + 2x - x^2$

b $y = x^2 + 9x$

c $y = x^3 - 27x + 9$

d $f(x) = x^3 - 6x^2 + 8$

e $f(x) = 3 + 9x - 3x^2 - x^3$

f $y = (x-1)(x^2-4)$

g $f(x) = x - 2\sqrt{x}, x \geq 0$

h $g(x) = x^4 - 8x^2 + 16$

i $y = (x-1)^2(x+1)$

3. Sketch the following functions:

a $y = 5 - 3x - x^2$

b $f(x) = x^2 + \frac{1}{2}x + \frac{3}{4}$

c $f(x) = x^3 + 6x^2 + 9x + 4$

d $f(x) = x^3 - 4x$

4. Find and describe the nature of all stationary points and points of inflection for the function $f(x) = x^3 + 3x^2 - 9x + 2$.

5. Sketch the graph of $x \mapsto x^4 - 4x^2$.

6. A function f is defined by $f: x \mapsto e^{-x} \sin x$, where $0 \leq x \leq 2\pi$.

a Find: i $f'(x)$ ii $f''(x)$

b Find the values of x for which:

i $f'(x) = 0$ ii $f''(x) = 0$.

c Using parts a and b, find the points of inflection and stationary points for f .

d Hence, sketch the graph of f .

7. A function f is defined by $f: x \mapsto e^x \sin x$, where $0 \leq x \leq 2\pi$.

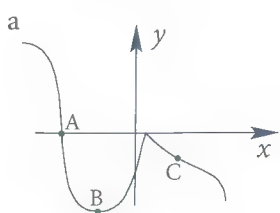
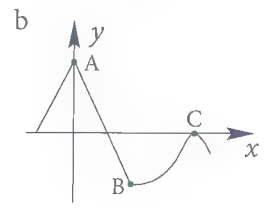
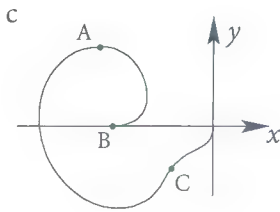
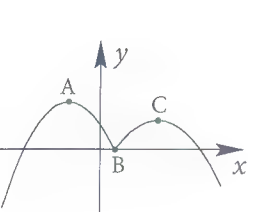
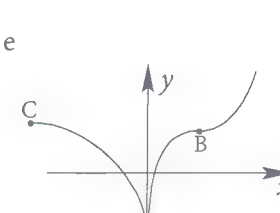
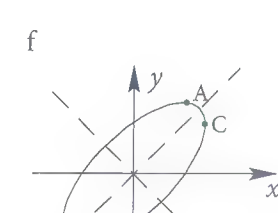
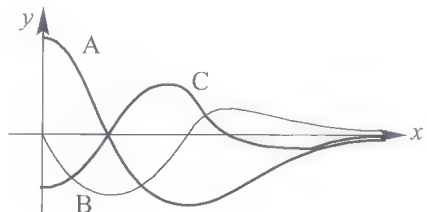
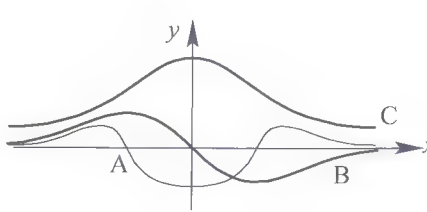
a Find: i $f'(x)$ ii $f''(x)$.

b Find the values of x for which:

i $f'(x) = 0$ ii $f''(x) = 0$.

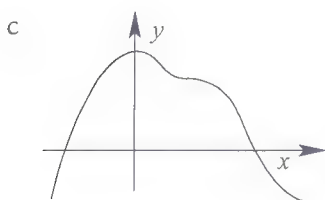
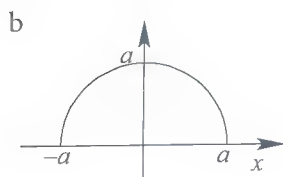
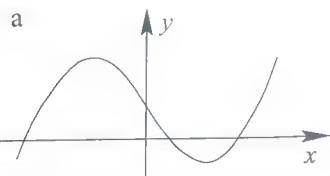
c Using parts a and b, find the points of inflection and stationary points for f .

d Hence, sketch the graph of f .

8. A function f is defined by $f: x \mapsto e^x \cos x$, where $0 \leq x \leq 2\pi$.
- Find: i $f'(x)$ ii $f''(x)$
 - Find the values of x for which:
 - $f'(x) = 0$
 - $f''(x) = 0$.
 - Using parts a and b, find the points of inflection and stationary points for f .
 - Hence, sketch the graph of f .
9. A function f is defined by $f: x \mapsto xe^{-x}$, where $x > 0$.
- Find: i $f'(x)$ ii $f''(x)$
 - Find the values of x for which:
 - $f'(x) = 0$
 - $f''(x) = 0$
 - Using parts a and b, find the points of inflection and stationary points for f .
 - Hence, sketch the graph of f .
- 10.
- Find the maximum value of the function $y = 6x - x^2$, $4 \leq x \leq 7$.
 - Find the minimum value of the function $y = 6x - x^2$, $2 \leq x \leq 6$.
 - Find the maximum value of the function $y = 2x - x^3$, $-2 \leq x \leq 6$.
 - Find the maximum value of the function $y = 36x - x^4$, $2 \leq x \leq 3$.
11. For the function $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 8$, $-6 \leq x \leq 6$, find:
- its minimum value.
 - its maximum value.
12. For each of the labelled points on the following graphs state:
- whether the derivative exists at the point.
 - the nature of the curve at the point.
- a 
- b 
- c 
- d 
- e 
- f 
13. Identify which graph corresponds to:
- $f(x)$
 - $f'(x)$
 - $f''(x)$
- a 
- b 

14. For each of the functions, $f(x)$, sketch:

i $f'(x)$ ii $f''(x)$



Answers



15. The curve with equation $y = ax^3 + bx^2 + cx + d$ has a local maximum where $x = -3$ and a local minimum where $x = -1$. If the curve passes through the points $(0, 4)$ and $(1, 20)$ sketch the curve for $x \in \mathbb{R}$.

16. The function $f(x) = ax^3 + bx^2 + cx + d$ has turning points at $(-1, -13/3)$ and $(3, -15)$. Sketch the graph of the curve $y = f(x)$.

17. The function $f: x \mapsto \mathbb{R}$, where $f(x) = ax^5 + bx^3 + cx$ has stationary points at $(-2, 64)$, $(2, -64)$ and $(0, 0)$. Find the values of a , b and c and hence sketch the graph of f .

18. Sketch the graph of the curve defined by the equation $y = x(10x - \ln x)$, $x > 0$ identifying, where they exist, all stationary points and points of inflection.

E.5 Kinematics

SL 5.9

This chapter deals with an application of calculus. Knowledge of standard derivatives and anti-derivatives is assumed. We suggest that you defer this chapter until you have studied the material in Chapter E.6 of this book.

The rate of change of distance with respect to time is known as velocity. Velocity can take both positive and negative values. Positive values are usually associated with motion up or to the right, negative values are associated with motion down or to the left.

Example E.5.1

An object moves along a straight horizontal track. Its position with respect to an origin is given by:

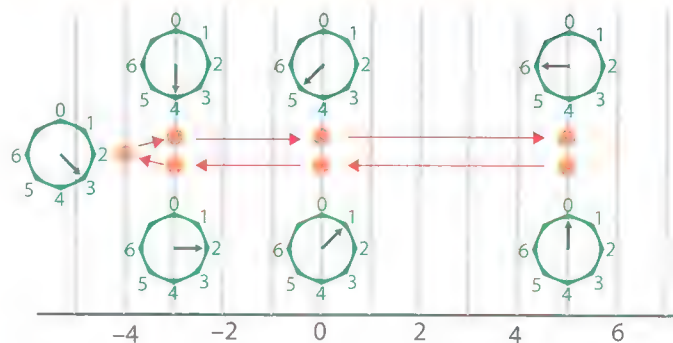
$$d(t) = t^2 - 6t + 5, 0 \leq t \leq 6$$

- Tabulate the position of the particle at times 0, 1, 2, 3, 4, 5 & 6.
- Show this as a diagram and a graph.
- Find $d'(t)$ and show the velocity information graphically.

- By direct substitution in the formula:

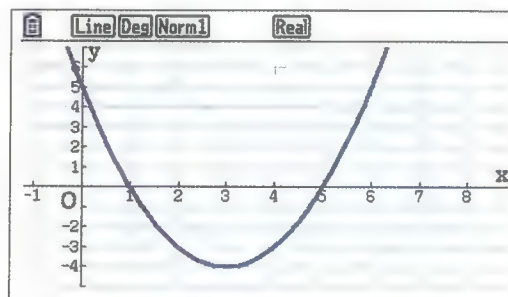
$$d(0) = 0^2 - 6 \times 0 + 5 = 5 \text{ etc.}$$

t	0	1	2	3	4	5	6
d	5	0	-3	-4	-3	0	5



The object initially moves to the left. At $t = 3$ it stops and reverses direction.

Graphically:



Note that, in this representation, displacement is displayed vertically.

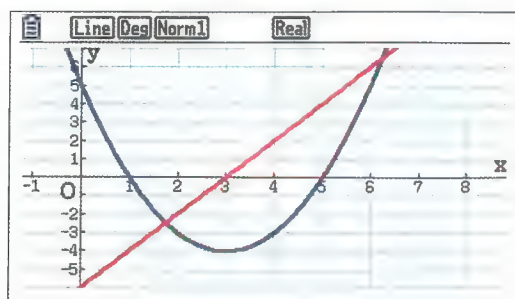
- Using the rules of differentiation:

$$d(t) = t^2 - 6t + 5, 0 \leq t \leq 6$$

$$d'(t) = 2t - 6, 0 \leq t \leq 6$$

t	0	1	2	3	4	5	6
d	5	0	-3	-4	-3	0	5
d'	-6	-4	-2	0	2	4	6

Graphically this is:



From $t = 0$ to 3, the velocity is negative and the object moves to the left, slowing down as it goes. From $t = 3$ to 6, the velocity is positive and the object moves to the right, speeding up as it goes.

The rate of change of velocity with respect to time is known as acceleration. Acceleration can take both positive and negative values. Positive values are usually associated with acceleration upwards or to the right, negative values are associated with acceleration downwards or to the left.

The difference between these three quantities is important and it is worth spending a bit of time getting them right.

If you are in a car:

Position or **distance** is measured by the odometer (or looking out the window).

Velocity is measured by the 'speedometer' - though car speedometers do not register speeds backwards ('in reverse').

Acceleration is the feeling of being pressed into the seat when the driver presses the accelerator pedal (positive) or being restrained by the seatbelt when the driver brakes (negative acceleration).

Knowledge of one of these quantities (and calculus) allows us to infer the other two.

Example E.5.2

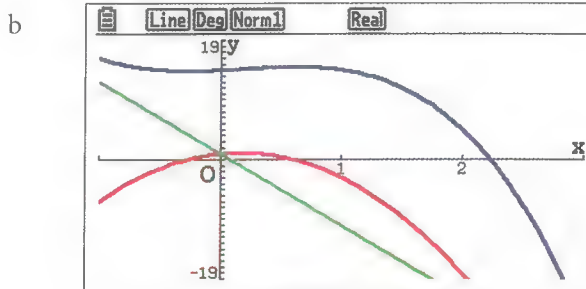
The height h metres above ground of a bumble bee at time t sec is modelled by: $h(t) = -2t^3 + t^2 + t + 15$, $0 \leq t \leq 2$.

- Find expressions for the velocity and acceleration of the bumble bee.
- Describe the motion and illustrate it graphically.

a $h(t) = -2t^3 + t^2 + t + 15$, $0 \leq t \leq 2$ (height)

$h'(t) = -6t^2 + 2t + 1$, $0 \leq t \leq 2$ (velocity)

$h''(t) = -12t + 1$, $0 \leq t \leq 2$ (acceleration)



The height graph (blue) shows an initial upward motion followed by an increasingly rapid descent. Do not confuse this graph with a map of the position of the bee. The horizontal scale is time, not horizontal position.

The velocity graph (red) shows an initial positive velocity - the bee is moving upwards. The velocity then becomes negative - downwards.

The acceleration graph (green) shows that there is an initial upward (+1) acceleration. This quickly becomes a larger and larger downward acceleration. The motion ends at time $t = 2$ with the bee at a height of 5 m.

Exercise E.5.1

- The distance (d metres) travelled along a horizontal track at time t is given by:

$d(t) = t^2 + 2t + 4$, $0 \leq t \leq 2$. Find:

- The initial position.
- The initial velocity.
- The acceleration.
- The highest velocity reached during the motion.

2. The height (h metres) reached by a firework at time t is given by:

$$h(t) = e^{0.5t} - 1, 0 \leq t \leq 10. \text{ Find:}$$

- Expressions for the velocity and acceleration.
- The maximum velocity achieved by the firework.

3. A particle's distance d from an origin at time t is given by:

$$h(t) = \sqrt{2t^2 - t + 1}, 0 \leq t \leq 5$$

Find:

- Expressions for the velocity and acceleration.
- When is the particle stationary?
- What is the largest speed reached by the particle?

4. The depth d (metres) of a probe designed to monitor subsurface currents t minutes after the start of an observation set is modelled by:

$$d(t) = \frac{t+1}{2t+1}$$

Find:

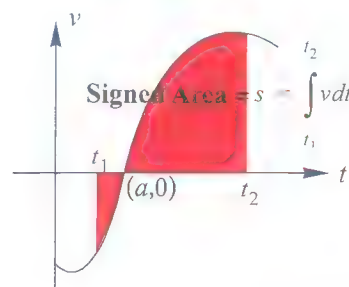
- Expressions for the velocity and acceleration.
- The vertical velocity when the depth is 0.75 m.

5. The table shows the displacement, d , of a particle at time t .

t	1	3	4	6
d	5	1	-4	-20

Find a quadratic polynomial that fits the data. Use the polynomial to find the velocity and acceleration of the particle at $t = 5$.

An application of integration when relating it to areas is that of kinematics. Just as the gradient of the displacement-time graph produces the velocity-time graph, so too then, we have that the **area beneath the velocity-time graph** produces the **displacement-time graph**. Notice that the area provides the displacement (not necessarily the distance!). Similarly with the acceleration-time graph, i.e. the **area under the acceleration-time graph** represents the **velocity**.



The displacement over the interval $[t_1, t_2]$ is given by

$$\text{Displacement} = s = \int_{t_1}^{t_2} v dt$$

However, the distance covered over the interval $[t_1, t_2]$ is given by

$$\text{Distance} = x = -\int_{t_1}^a v dt + \int_a^{t_2} v dt$$

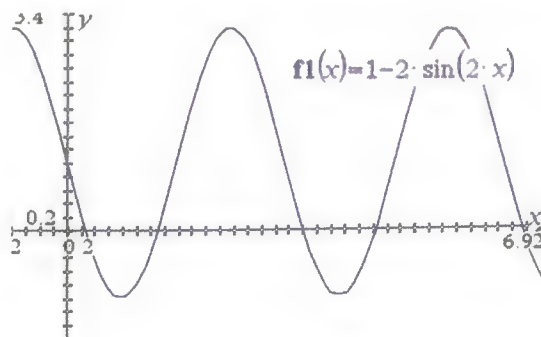
Example E.5.3

The velocity of an object is v m/s after t seconds, where $v = 1 - 2 \sin 2t$.

Find the object's displacement over the first $\frac{3\pi}{4}$ seconds.

Find the distance travelled by the object over the first $\frac{3\pi}{4}$ seconds.

It is always a good idea to sketch the velocity-time graph:



The displacement is then given by:

$$s = \int_0^{\frac{3\pi}{4}} (1 - 2 \sin 2t) dt = \left[t + \cos 2t \right]_0^{\frac{3\pi}{4}}$$

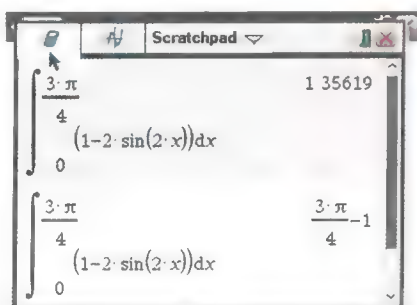
$$= \left(\frac{3\pi}{4} + \cos\left(\frac{6\pi}{4}\right) \right) - (0 + \cos(0))$$

$$= \frac{3\pi}{4} - 1$$

$$\approx 1.36$$

That is, the object's displacement measures (approx.) 1.36 metres.

This time, we will use the scratchpad of the TI calculator followed by MENU 4 Calculus 3 Integral.



The required integral can be entered and either evaluated exactly or approximately.

As part of the graph lies below the t -axis, when determining the distance travelled we use the same principle as that which differentiates between the signed area and the actual area enclosed by a curve and the horizontal axis. In short:

Displacement = Signed area

Distance = Area.

The first step is to determine the t -intercepts:

Solving for $1 - 2 \sin 2t = 0$ we have:

$$\sin 2t = \frac{1}{2} \Rightarrow 2t = \frac{\pi}{6}, \frac{5\pi}{6}, \dots \therefore t = \frac{\pi}{12}, \frac{5\pi}{12}, \dots$$

Note that we only require the first two intercepts. Therefore, the distance is given by:

$$x = \int_0^{\frac{\pi}{12}} (1 - 2 \sin 2t) dt - \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} (1 - 2 \sin 2t) dt + \int_{\frac{5\pi}{12}}^{\frac{3\pi}{4}} (1 - 2 \sin 2t) dt$$

Evaluating this expression is rather lengthy, and—unless we require an exact value—we might as well make use of the graphics calculator. There are a number of ways this can be

done. We will use absolute value applied to the function.

$$\int_0^{\frac{3\pi}{4}} \frac{1}{4} |1 - 2 \sin(2 \cdot x)| dx \quad 2.7259$$

Distance travelled: $= 0.1278 - (-0.6849) + 1.9132 = 2.7259$.

That is, object travelled (approx.) 2.73 m.

Example E.5.4

A rocket starts from rest and accelerates such that its acceleration is given by the formula $a(t) = 3t + t^2$, $0 \leq t \leq 10$ where distances are measured in metres and time in seconds. Find the distance travelled in the first ten seconds of the rocket's motion.

The information is given as an acceleration. We must find the indefinite integral of this function to get a rule to give us the velocity.

$$a(t) = 3t + t^2, 0 \leq t \leq 10 \Rightarrow v(t) = \int (3t + t^2) dt = \frac{3t^2}{2} + \frac{t^3}{3} + c$$

Now, when $t = 0$, $v = 0 \therefore 0 = 0 + 0 + c \Rightarrow c = 0$

The constant is zero because we are told that the rocket starts from rest. The distance travelled is the area under this velocity time graph. This must be found using definite integration. As the graph of $v(t)$ lies above the t -axis over the interval $0 \leq t \leq 10$, we have the distance D , given by

$$D = \int_0^{10} \left(\frac{3t^2}{2} + \frac{t^3}{3} \right) dt = \left[\frac{t^3}{2} + \frac{t^4}{12} \right]_0^{10}$$

$$= \frac{10^3}{2} + \frac{10^4}{12}$$

$$= 1333 \frac{1}{3} \text{ m}$$

The technique described in this example is the basis of the inertial navigator. This senses acceleration and integrates it to infer velocity. The instrument then integrates a second time to calculate distance travelled. Of course, none of these quantities are generally expressed as exact mathematical formulae and the calculation has to be performed using numerical approximation.

Exercise E.5.2

1. Find the displacement equation, $x(t)$, for each of the following:
 - a $\frac{d^2x}{dt^2} = 6t$ where $\frac{dx}{dt} = 3$ and $x = 10$ when $t = 0$.
 - b $\frac{d^2x}{dt^2} = -(4\sin t + 3\cos t)$ where $\frac{dx}{dt} = 4$ and $x = 2$ when $t = 0$.
 - c $\frac{d^2x}{dt^2} = 2 - e^{-\frac{1}{2}t}$ where $\frac{dx}{dt} = 4$ and $x = 0$ when $t = 0$.
2. The acceleration, $a(t) \text{ ms}^{-2}$, of a body travelling in a straight line and having a displacement $x(t) \text{ m}$ from an origin is governed by $a(t) = 6t - 2$ where:

$$\frac{dx}{dt} = 0 \text{ and } x = 0 \text{ when } t = 0.$$
 - a Find the displacement of the body at any time t .
 - b Find the displacement of the body after 5 seconds.
 - c Find the distance the body has travelled after 5 seconds.
3. A body moves along a straight line in such a way that its velocity, $v \text{ ms}^{-1}$, is given by $v(t) = -\sqrt{t+4} + 2$. After 5 seconds of motion the body is at the origin O.
 - a Sketch the displacement-time graph for this body.
 - b How far will the body have travelled after another 5 seconds.
4. A particle starts from rest and moves with a velocity, $v \text{ ms}^{-1}$, where $v = t(t-5)$. Find the distance travelled between the two occasions when the particle is at rest.
5. A stone is thrown vertically upwards from ground level with a velocity of 25 ms^{-1} . If the acceleration of the stone is 9.8 ms^{-2} directed downwards, find the time taken before the stone reaches its highest point and the total distance travelled when the stone falls back to the ground.
6. The velocity of a particle is given by $v(t) = 3 - 3\sin 3t$ which is measured in m/s.
 - a Find when the particle first comes to rest.
 - b Find the distance travelled by the particle from when it started to when it first comes to rest.
7. An object, starting from rest, moves in a straight line with an acceleration that is given by:

$$a(t) = \frac{12}{(t+1)^2} \text{ ms}^{-2}.$$
 Find the distance travelled during the first 9 seconds.
8. An object has its velocity governed by the equation:

$$v(t) = 10\sin\left(\frac{\pi}{16}t\right) \text{ m/s}.$$
 - a Given that $s(0) = 0$, find its displacement equation.
 - b Find its displacement after 20 seconds.
 - c Find its displacement during the 20th second.
 - d How far has it travelled in twenty seconds?
9. A particle moving in a straight line has its acceleration, $a \text{ ms}^{-2}$, defined by the equation:

$$a = \frac{2k}{t^3} \text{ ms}^{-2}.$$
 At the end of the first second of motion, the particle has a velocity measuring 4 ms^{-1} .
 - a Find an expression for the velocity of the particle.
 - b Given that its velocity approaches a limiting value of 6 ms^{-1} , find k .
 - c Find the distance travelled by the particle after a further 9 seconds.
10.
 - a Show that:

$$\frac{d}{dx}\left[\frac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx)\right] = e^{ax}\cos bx.$$

- b The velocity of a vibrating bridge component is modelled by the function $V = e^{-2t}\cos(3t)$. V ms^{-1} is the velocity of the component and t is the time in seconds after the observations begin.

Find the distance the component travels in the first tenth of a second.

Answers



Calculus

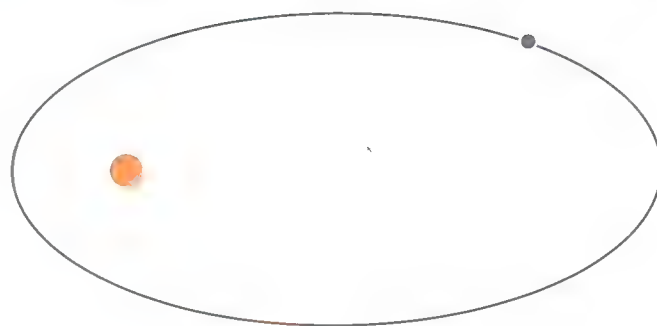
The word 'calculus' is derived from the Latin word meaning 'stone'. The connection between stones and calculation is the stone abacus:



However, the problem that set Isaac Newton thinking was astronomic. The story that it was a falling apple is, however, Newton's.

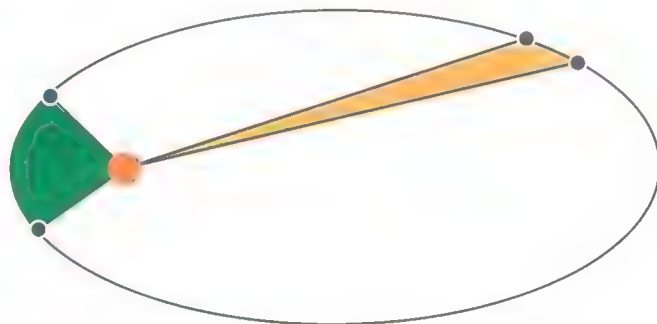
Actually, the problem was the details of planetary motion.

The German astronomer Johannes Kepler (1571-1630) had discovered that the planets move around the sun in elliptical orbits. The sun is at one of the foci of the ellipse.



Kepler had discovered more than that. The planet moves more slowly as its distance from the sun increases,

Specifically, the area it sweeps out in a given period is the same wherever it is in its orbit.



The green and orange areas are the same because the planet moves faster near the sun (green area) than it does at greater distances (orange area).

Animation of planetary motion



Newton was looking at a very complex problem as everything appeared to be continuously changing.

This was the stimulus to develop the calculus - the study of variation.

His conclusion was that the force between the two bodies was proportional to the masses of the bodies and inversely proportional to the square of the distance between them.

A truly remarkable intellectual achievement.

The Passage from the Brazils to the Cape of Good Hope ; with an Account of the Transactions of the Fleet there.

E.6 Integration.

September 1787.

SL 5.10
SL 5.11
OUR passage from Rio de Janeiro to the Cape of Good Hope, was equally prosperous with that which had preceded

In Chapter E.6 of the common Core Text, we introduced the idea of antidifferentiation - the reverse process to differentiation. The technique was illustrated with examples restricted to polynomials. To summarise the rules covered:

$$\text{Differentiation: } y = ax^n \Rightarrow \frac{dy}{dx} = anx^{n-1}$$

also, sums and differences of terms can be differentiated term by term:

$$y = ax^n + bx^m \Rightarrow \frac{dy}{dx} = anx^{n-1} + bmx^{m-1}$$

Recall also that there are two notations in common use and students should be familiar with both. The above notation is due to Gottfried Wilhelm Leibniz (1646-1716) and bears his name.

The other notation is due to Isaac Newton (1642-1727) and bears his name. In Newton's notation, these two rules are:

$$\text{Differentiation: } f(x) = ax^n \Rightarrow f'(x) = anx^{n-1} \text{ and}$$

$$f(x) = ax^n + bx^m \Rightarrow f'(x) = anx^{n-1} + bmx^{m-1}.$$

The rules running in reverse (antidifferentiation) are:

$$\frac{dy}{dx} = anx^{n-1} \Rightarrow y = ax^n \text{ and } f'(x) = anx^{n-1} \Rightarrow f(x) = ax^n.$$

It is, of course, much more convenient to have a rule for x^n .

$$\frac{dy}{dx} = ax^n \Rightarrow y = \frac{a}{n+1} x^{n+1}$$

You should check that the expression $y = \frac{a}{n+1} x^{n+1}$ does

$$\text{differentiate to } \frac{dy}{dx} = ax^n$$

Polynomials can be differentiated term by term. The same applies to antidifferentiation.

An additional, and very common notation for antidifferentiation is:

$$\int ax^n dx = \frac{a}{n+1} x^{n+1}$$

The significance of the ' dx ' is to record that x is the variable and all other pronumerals are to be regarded as constants. The rules of calculus treat the two categories completely differently.

One final factor to consider is that the derivatives of constants are zero. When reversing the process, we should always add an arbitrary constant (often ' c '). Thus the full rule for antidifferentiating a polynomial term is:

$$\int ax^n dx = \frac{a}{n+1} x^{n+1} + c$$

The integral sign is an archaic printer's 'S' - see our header picture.

There are several forms of words that also require antidifferentiation. For example:

"The gradient of a curve is... Find the equation of the curve".

"The velocity of a particle is... Find its position.

It is also the case that the term 'integration' means the same as antidifferentiation.

Example E.6.1

1. Antidifferentiate with respect to x : $2x^3 + 3x - 1$.
2. Find y if $\frac{dy}{dx} = 5x^2 + 1$.
3. If $f'(x) = \frac{1}{x^2}$, find $f(x)$.
4. Evaluate: $\int (x^4 - 3) dx$.

1. Using the rules just discussed, the antiderivative is:

$$\frac{2}{3+1}x^{3+1} + \frac{3}{1+1}x^{1+1} - \frac{1}{0+1}x^{0+1} + c = \frac{x^4}{2} + \frac{3}{2}x^2 - x + c$$

The third term results from the fact that $x^0 = 1$ and the instruction is to antidifferentiate with respect to x .

2. Using the rule:

$$\frac{dy}{dx} = 5x^2 + 1 \Rightarrow y = \frac{5}{3}x^3 + x + c$$

$$3. \quad f'(x) = \frac{1}{x^2} = x^{-2} \Rightarrow f(x) = \frac{1}{-2+1}x^{-2+1} = \frac{-1}{x} + c$$

$$4. \quad \int (x^4 - 3) dx = \frac{x^5}{5} - 3x + c$$

So far, we have only used this 'powers rule' with integer indices. The rule does, however, apply to all real indices:

Example E.6.2

1. Antidifferentiate with respect to x : $2x^{-3} - 2$.
2. Find y if $\frac{dy}{dx} = \sqrt{2x} - 3$.
3. If $f'(x) = (2x - \sqrt{x})^2$, find $f(x)$.
4. Evaluate: $\int \frac{\sqrt{x}+1}{x^2} dx$.

$$1. \quad \int (2x^{-3} - 2) dx = \frac{2}{-3+1}x^{-3+1} - 2x + c = -\frac{1}{x^2} - 2x + c$$

2. It is always a good idea to write surds as powers of x before applying the rules.

$$\frac{dy}{dx} = \sqrt{2x} - 3 \Rightarrow \frac{dy}{dx} = \sqrt{2}x^{1/2} - 3x^0$$

Next, use the rules:

$$y = \frac{\sqrt{2}}{1/2+1}x^{1/2+1} - \frac{3}{0+1}x^{0+1} + c = \frac{2\sqrt{2}}{3}x\sqrt{x} - 3x + c$$

3. First write as powers:

$$f'(x) = (2x - \sqrt{x})^2 = 4x^2 - 4x^{3/2} + x$$

Then apply the rules:

$$f(x) = \frac{4}{2+1}x^{2+1} - \frac{4}{3/2+1}x^{3/2+1} + \frac{x^2}{2} + c$$

$$= \frac{4}{3}x^3 - \frac{8}{5}x^{5/2} + \frac{x^2}{2} + c$$

$$= \frac{4}{3}x^3 - \frac{8}{5}x^{5/2} + \frac{x^2}{2} + c$$

$$= \frac{4}{3}x^3 - \frac{8}{5}x^2\sqrt{x} + \frac{x^2}{2} + c$$

$$4. \quad \int \frac{\sqrt{x}+1}{x^2} dx = \int (x^{1/2-2} + x^{-2}) dx$$

$$= \int (x^{-3/2} + x^{-2}) dx$$

$$= \frac{1}{-3/2+1}x^{-3/2+1} + \frac{1}{-2+1}x^{-2+1} + c$$

$$= -2x^{-1/2} - x^{-1} + c$$

$$= -\frac{2}{\sqrt{x}} - \frac{1}{x} + c$$

Exercise E.6.1

1. Antidifferentiate with respect to x :

$$a \quad 4x^3 - x + 1$$

$$b \quad x\sqrt{x} - 2$$

$$c \quad \left(x + \frac{1}{x}\right)^2$$

$$d \quad \sqrt{x^5}$$

$$e \quad \frac{3}{\sqrt{x}}$$

$$f \quad (2x-1)(3x+1)$$

2. Find y if

a $\frac{dy}{dx} = \frac{6\sqrt{x}}{x}$

b $\frac{dy}{dx} = \left(\frac{1}{\sqrt{x}} + 6\right)$

c $\frac{dy}{dx} = \left(1 + \frac{1}{x^2}\right)^2$

d $\frac{dy}{dx} = \sqrt[3]{x}$

e $\frac{dy}{dx} = (\sqrt[3]{x} - 1)^2$

f $\frac{dy}{dx} = \frac{7}{x^6}$

3. Evaluate:

a $\int \sqrt[3]{x^2} dx$

b $\int (\sqrt[3]{x})^4 dx$

c $\int (\sqrt[3]{8x} + 2)^2 dx$

d $\int (x+1)^4 dx$

e $\int (2x-1)^4 dx$

f $\int (\sqrt[5]{x} - \sqrt[3]{x}) dx$

More Standard Integrals

In the same way that there are rules for differentiating functions other than those of the form ax^n , we also have standard rules for integrating functions other than the ones that we have dealt with so far. That is, there are standard rules for finding the indefinite integral of circular trigonometric functions, exponential functions and logarithmic functions.

We can deduce many such antiderivatives using the result $\int \frac{d}{dx}(h(x)) dx = h(x) + c - (1)$.

For example, if we consider the derivative $\frac{d}{dx}(e^{2x}) = 2e^{2x}$,

We can write $\int \frac{d}{dx}(e^{2x}) dx = \int 2e^{2x} dx$.

But from (1) we have $\int \frac{d}{dx}(e^{2x}) dx = e^{2x}$.

Therefore, $e^{2x} + c = \int 2e^{2x} dx$.

Or, we could write:

$$e^{2x} + c = 2 \int e^{2x} dx \Leftrightarrow \int e^{2x} dx = \frac{1}{2}(e^{2x} + c) = \frac{1}{2}e^{2x} + c$$

Similarly, $\frac{d}{dx}(\ln x) = \frac{1}{x}$ and so, antidifferentiating both sides

we have $\int \frac{d}{dx}(\ln x) dx = \int \frac{1}{x} dx$.

Then, as $\int \frac{d}{dx}(\ln x) dx = \ln x$, we can write, $\ln x + c = \int \frac{1}{x} dx$.

We summarize these rules in the table below:

$f(x)$	$\int f(x) dx$	Examples
$ax^n, n \neq -1$	$\frac{a}{n+1} x^{n+1} + c$ $n \neq -1$	$\int 2x^2 dx = \frac{2}{3}x^3 + c$
$\frac{1}{x}, x \neq 0$	$\log_e x + c, x \neq 0$ or $\log_e x + c, x > 0$	$\int \frac{6}{x} dx = 6 \int \frac{1}{x} dx$ $= 6 \log_e x + c, x > 0$
$\sin(kx)$	$-\frac{1}{k} \cos(kx) + c$	$\int \sin(5x) dx$ $= -\frac{1}{5} \cos(5x) + c$
$\cos(kx)$	$\frac{1}{k} \sin(kx) + c$	$\int \cos\left(\frac{x}{4}\right) dx = \frac{1}{\left(\frac{1}{4}\right)} \sin\left(\frac{x}{4}\right) + c$ $= 4 \sin\left(\frac{x}{4}\right) + c$
$\sec^2(kx)$	$\frac{1}{k} \tan(kx) + c$	$\int \sec^2(2x) dx = \frac{1}{2} \tan(2x) + c$
e^{kx}	$\frac{1}{k} e^{kx} + c$	$\int e^{-3x} dx = -\frac{1}{3} e^{-3x} + c$

One of the cases we have been avoiding is:

$$\int \frac{1}{x} dx = \int x^{-1} dx = \frac{1}{-1+1} x^{-1+1} + c$$

This implies division by zero and so cannot be correct!

NB: In the previous table (and from here on), the constant c is assumed to be a real number.

General power rule

The indefinite integral of $f(ax + b)$:

$$\text{If } \int f(x) dx = F(x) + c, \text{ then } \int f(ax + b) dx = \frac{1}{a} F(ax + b) + c.$$

This means that all of the integrals in the table can be generalized further.

In particular we consider the generalized power rules:

$$\int (ax + b)^n dx = \frac{1}{a(n+1)} (ax + b)^{n+1} + c, n \neq -1$$

$$\frac{1}{ax + b} dx = \frac{1}{a} \log_e(ax + b) + c, ax + b > 0$$

Example E.6.3

Find the antiderivative of:

a $(3x + 7)^4$ b $3(4 - 2x)^5$

c $\sqrt{5x + 1}$ d $\frac{4}{(2x - 3)^2}$

This question requires the use of the result:

$$\int (ax + b)^n dx = \frac{1}{a(n+1)} (ax + b)^{n+1} + c, n \neq -1$$

a
$$\int (3x + 7)^4 dx = \frac{1}{3 \times (4 + 1)} (3x + 7)^{4+1} + c$$

$$= \frac{1}{15} (3x + 7)^5 + c$$

b
$$\int 3(4 - 2x)^5 dx = 3 \int (4 - 2x)^5 dx$$

$$= 3 \times \frac{1}{(-2) \times (5 + 1)} (4 - 2x)^{5+1} + c$$

$$= \frac{3}{-2 \times 6} (4 - 2x)^6 + c$$

$$= -\frac{1}{4} (4 - 2x)^6 + c$$

c
$$\int \sqrt{5x + 1} dx = \int (5x + 1)^{1/2} dx$$

(Note: we must first convert into power form!)

$$= \frac{1}{5 \times \left(\frac{1}{2} + 1\right)} (5x + 1)^{\frac{1}{2} + 1} + c$$

$$= \frac{2}{15} (5x + 1)^{\frac{3}{2}} + c$$

d
$$\int \frac{4}{(2x - 3)^2} dx = 4 \int (2x - 3)^{-2} dx$$

$$= 4 \times \frac{1}{2 \times (-2 + 1)} (2x - 3)^{-2+1} + c$$

$$= -2 \times (2x - 3)^{-1} + c$$

$$= -\frac{2}{(2x - 3)} + c$$

Example E.6.4

Find the antiderivative of:

a $\sin 4x$ b $\cos\left(\frac{1}{2}x\right)$ c $\sec^2(3x)$.

a
$$\int \sin 4x dx = -\frac{1}{4} \cos 4x + c$$

 [Using $\int \sin(kx) dx = -\frac{1}{k} \cos(kx) + c$]

b
$$\int \cos\left(\frac{1}{2}x\right) dx = \frac{1}{\left(\frac{1}{2}\right)} \sin\left(\frac{1}{2}x\right) + c = 2 \sin\left(\frac{1}{2}x\right) + c$$

 [Using $\int \cos(kx) dx = \frac{1}{k} \sin(kx) + c$]

c
$$\int \sec^2(3x) dx = \frac{1}{3} \tan(3x) + c$$

 [Using $\int \sec^2(kx) dx = \frac{1}{k} \tan(kx) + c$]

We can extend the results for the antiderivatives of circular trigonometric functions as follows:

$f(x)$	$\int f(x) dx$
$\sin(ax + b)$	$-\frac{1}{a} \cos(ax + b) + c$
$\cos(ax + b)$	$\frac{1}{a} \sin(ax + b) + c$
$\sec^2(ax + b)$	$\frac{1}{a} \tan(ax + b) + c$

Example E.6.5

Find the antiderivative of:

a e^{2x}

b $5e^{-3x}$

c $4x - 2e^{\frac{1}{3}x}$

d $\frac{5}{x}$

e $\frac{2}{3x+1}$

a $\int e^{2x} dx = \frac{1}{2}e^{2x} + c$. (Using $\int e^{kx} dx = \frac{1}{k}e^{kx} + c$,

b
$$\begin{aligned}\int 5e^{-3x} dx &= 5 \int e^{-3x} dx \\ &= 5 \times \frac{1}{-3} e^{-3x} + c \\ &= -\frac{5}{3} e^{-3x} + c \\ &\quad \left(\text{Using } \int e^{kx} dx = \frac{1}{k} e^{kx} + c \right)\end{aligned}$$

c
$$\begin{aligned}\int \left(4x - 2e^{\frac{1}{3}x} \right) dx &= \int 4x dx - 2 \int e^{\frac{1}{3}x} dx \\ &= 4 \times \frac{1}{2} x^2 - 2 \times \frac{1}{\left(\frac{1}{3}\right)} e^{\frac{1}{3}x} + c \\ &= 2x^2 - 6e^{\frac{1}{3}x} + c\end{aligned}$$

d $\int \frac{5}{x} dx = 5 \int \frac{1}{x} dx = 5 \ln(x) + c, x > 0$

e
$$\begin{aligned}\int \frac{2}{3x+1} dx &= 2 \int \frac{1}{3x+1} dx = \frac{2}{3} \ln(3x+1) + c, x > -\frac{1}{3} \\ &\quad \left[\int \frac{1}{ax+b} dx = \frac{1}{a} \log_e(ax+b) + c \right]\end{aligned}$$

As we have done for the circular trigonometric functions, we can extend the antiderivative of the exponential function to:

$$e^{ax+b} dx = \frac{1}{a} e^{ax+b} + c$$

Example E.6.6

a $\int 2(4x-1)^6 dx$ b $\int \frac{7}{4-3x} dx$

c $\int \frac{3}{\sqrt{2x+1}} dx$ d $\int e^{-2x+3} dx$

e $\int (e^{-4x} + \sqrt[5]{3x+2}) dx$

f $\int \left(\cos\left(\frac{\pi}{2}x\right) - \frac{2}{6x+5} \right) dx$

a We have $n = 6$, $a = 4$ and $b = -1$,

$$\begin{aligned}\therefore \int 2(4x-1)^6 dx &= \frac{2}{4(6+1)} (4x-1)^{6+1} + c \\ &= \frac{1}{14} (4x-1)^7 + c\end{aligned}$$

b This time we use the case for $n = -1$, with $a = -3$ and $b = 4$, giving:

$$\int \frac{7}{4-3x} dx = 7 \int \frac{1}{4-3x} dx = -\frac{7}{3} \log_e(4-3x) + c, x < \frac{4}{3}$$

c We first rewrite the indefinite integral in power form:

$$\int \frac{3}{\sqrt{2x+1}} dx = \int 3(2x+1)^{-\frac{1}{2}} dx$$

So that $a = 2$, $b = 1$ and $n = -\frac{1}{2}$.

Therefore we have that:

$$\begin{aligned}\int \frac{3}{\sqrt{2x+1}} dx &= \int 3(2x+1)^{-\frac{1}{2}} dx \\ &= \frac{3}{2\left(-\frac{1}{2}+1\right)} (2x+1)^{-\frac{1}{2}+1} + c \\ &= \frac{3}{2 \times \frac{1}{2}} (2x+1)^{\frac{1}{2}} + c \\ &= 3\sqrt{2x+1} + c\end{aligned}$$

$$d \quad \int e^{-2x+3} dx = -\frac{1}{2}e^{-2x+3} + c$$

$$e \quad \int (e^{-4x} + \sqrt[5]{3x+2}) dx = \int \left(e^{-4x} + (3x+2)^{\frac{1}{5}} \right) dx$$

$$= -\frac{1}{4}e^{-4x} + \frac{1}{3\left(\frac{6}{5}\right)}(3x+2)^{\frac{6}{5}} + c$$

$$= -\frac{1}{4}e^{-4x} + \frac{5}{18}\sqrt[5]{(3x+2)^6} + c$$

$$f \quad \int \left(\cos\left(\frac{\pi}{2}x\right) - \frac{2}{6x+5} \right) dx = \frac{1}{\frac{\pi}{2}} \sin\left(\frac{\pi}{2}x\right) - \frac{2}{6} \ln(6x+5) + c$$

$$= \frac{2}{\pi} \sin\left(\frac{\pi}{2}x\right) - \frac{1}{3} \ln(6x+5) + c$$

Exercise E.6.2

1. Find the indefinite integral of:

a e^{5x}	b e^{3x}	c e^{2x}
d $e^{0.1x}$	e e^{-4x}	f $4e^{-4x}$
g $0.1e^{-0.5x}$	h $2e^{1-x}$	i $5e^{x+1}$
j $-2e^{2-2x}$	k $e^{\frac{1}{3}x}$	l $\sqrt{e^x}$

2. Find the indefinite integral of:

a $\frac{4}{x}$	b $-\frac{3}{x}$	c $\frac{2}{5x}$
d $\frac{1}{x+1}$	e $\frac{1}{2x}$	
f $\left(1 - \frac{1}{x}\right)^2$	g $\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2$	
h $\frac{3}{x+2}$		

3. Find the indefinite integral of:

a $\sin(3x)$	b $\cos(2x)$
c $\sec^2(5x)$	d $\sin(-x)$

4. Find the indefinite integral of:

a $\sin(2x) + x$	b $6x^2 - \cos(4x)$
c e^{5x}	d $4e^{-3x} + \sin\left(\frac{1}{2}x\right)$
e $\cos\left(\frac{x}{3}\right) - \sin(3x)$	f $e^{2x} + \frac{4}{x} - 1$
g $(e^x + 1)^2$	
h $-5\sin(4x) + \frac{x-1}{x}$	
i $\sec^2(3x) - \frac{2}{x} + e^{\frac{1}{2}x}$	
j $(e^x - e^{-x})^2$	k e^{2x+3}
l $\sin(2x + \pi)$	m $\cos(\pi - x)$
n $\sin\left(\frac{1}{4}x + \frac{\pi}{2}\right)$	o $\frac{e^x - 2}{\sqrt{e^x}}$

5. Using the general power rule, find the indefinite integral of:

a $(4x-1)^3$	b $(3x+5)^6$
c $(2-x)^4$	d $(2x+3)^5$
e $(7-3x)^8$	f $\left(\frac{1}{2}x-2\right)^9$
g $(5x+2)^{-6}$	h $(9-4x)^{-2}$
i $(x+3)^{-3}$	j $\frac{1}{x+1}$
k $\frac{2}{2x+1}$	l $\frac{4}{3-2x}$
m $\frac{-3}{5-x}$	n $\frac{9}{3-6x}$
o $\frac{5}{3x+2}$	

6. Find the antiderivative of:

a $\sin(2x-3) - 2x$
b $3\cos\left(2 + \frac{1}{2}x\right) + 5$
c $\frac{1}{2}\cos\left(2 - \frac{1}{3}x\right) - \frac{2}{2x+1}$
d $\sec^2(0.1x-5) - 2$
e $\frac{4}{2x+3} - e^{-\frac{1}{2}x+2}$

$$f \quad \frac{4}{(2x+3)^2} - e^{2x-\frac{1}{2}}$$

$$g \quad \frac{x+2}{x+1} - \frac{4}{x+2}$$

$$h \quad \frac{2x+1}{x+2} + \frac{1}{2x+1}$$

$$i \quad \frac{2}{(2x+1)^2} + \frac{2}{2x+1}$$

7. Find $f(x)$ given that:

$$a \quad f'(x) = \sqrt{4x+5} \text{ where } f(-1) = \frac{1}{6}$$

$$b \quad f'(x) = \frac{8}{4x-3} \text{ where } f(1) = 2.$$

$$c \quad f'(x) = \cos(2x+3) \text{ where } f\left(\pi - \frac{3}{2}\right) = 1$$

$$d \quad f'(x) = 2 - e^{-2x+1} \text{ where } f(0) = e.$$

8. A bacteria population, N thousand, has a growth rate modelled by the equation:

$$\frac{dN}{dt} = \frac{4000}{1+0.5t}, \quad t \geq 0$$

where t is measured in days. Initially there are 250 bacteria in the population.

Find the population size after 10 days.

Substitution Rule

In the previous section, we considered integrals that required the integrand to be of a particular form in order to carry out the antidifferentiation process.

For example, the integral $\int 2x\sqrt{1+x^2} dx$ is of the form $\int h'(x)[h(x)]^n dx$ and so we could proceed by using the result:

$$\int h'(x)[h(x)]^n dx = \frac{1}{n+1}[h(x)]^{n+1} + c.$$

Next consider the integral $\int x\sqrt{x-1} dx$. This is not in the form $\int h'(x)[h(x)]^n dx$ and so we cannot rely on the recognition approach we have used so far. To determine such an integral we need to use a formal approach.

Indefinite integrals that require the use of the general power rule can also be determined by making use of a method known as the substitution rule (or change of variable rule). The name of the rule is indicative of the process itself. We introduce a new variable, u (say), and substitute it for an appropriate part (or the whole) of the integrand. An important feature of this method is that it will enable us to find the integral of expressions that cannot be determined by the use of the general power rule.

We illustrate this process using a number of examples (remembering that the success of this method is in making the appropriate substitution). The basic steps in integration by substitution can be summarized as follows:

1. Define u (i.e. let u be a function of the variable which is part of the integrand).
2. Convert the integrand from an expression in the original variable to an expression in u (this means that you also need to convert the ' dx ' term to a ' du ' term – if the original variable is x).
3. Integrate and then rewrite the answer in terms of x (by substituting back for u).

NB: This is only a guide, you may very well skip steps or use a slightly different approach.

Extra questions



Example E.6.7

Find:

a $\int (2x + 1)^4 dx$

b $\int 2x(x^2 + 1)^3 dx$ c $\int \frac{x^2}{\sqrt{x^3 - 4}} dx$.

a Although this integral can be evaluated by making use of the general power rule, we use the substitution method to illustrate the process:

In this case we let $u = 2x + 1 \Rightarrow \frac{du}{dx} = 2 \therefore dx = \frac{1}{2} du$.

Having chosen u , we have also obtained an expression for dx and we are now in a position to carry out the substitution for the integrand:

$$\begin{aligned} \int (2x + 1)^4 dx &= \int u^4 \times \left(\frac{1}{2} du\right) = \frac{1}{2} \int u^4 du = \frac{1}{2} \times \frac{1}{5} u^5 + c \\ &= \frac{1}{10} u^5 + c \end{aligned}$$

Substituting back, we obtain, in terms of x : $= \frac{1}{10} (2x + 1)^5 + c$

b This time, we let $u = x^2 + 1$. Note the difference between this substitution and the one used in part a. We are making a substitution for a non-linear term!

Now, $u = x^2 + 1 \Rightarrow \frac{du}{dx} = 2x \therefore \frac{1}{2} du = x dx$.

Although there is an x attached to the dx term, hopefully, when we carry out the substitution, everything will fall into place.

Now, $\int 2x(x^2 + 1)^3 dx = \int 2(x^2 + 1)^3 x dx$

(We have moved the x next to the dx .)

$$\begin{aligned} &= \int 2u^3 \times \frac{1}{2} du \text{ (substituting } x dx \text{ for } \frac{1}{2} du \text{.)} \\ &= \frac{1}{4} u^4 + c \\ &= \frac{1}{4} (x^2 + 1)^4 + c \end{aligned}$$

NB: A second (alternate) method is to obtain an expression for dx in terms of one or both variables. Make the substitution and then simplify. Although there is some dispute as to the 'validity' of this method, in essence it is the same. We illustrate this now:

c Let $u = x^3 - 4 \Rightarrow \frac{du}{dx} = 3x^2 \therefore dx = \frac{1}{3x^2} du$, making

the substitution for u and dx , we have:

$$\begin{aligned} \int \frac{x^2}{\sqrt{x^3 - 4}} dx &= \int \frac{x^2}{\sqrt{u}} \times \frac{1}{3x^2} du = \frac{1}{3} \int u^{-1/2} du \text{ (Notice the } x^2 \text{ terms cancel!)} \\ &= \frac{2}{3} u^{1/2} + c \\ &= \frac{2}{3} \sqrt{x^3 - 4} + c \end{aligned}$$

Example E.6.8

Find $\int x \sqrt{x - 1} dx$.

Letting $u = x - 1 \Rightarrow \frac{du}{dx} = 1 \therefore du = dx$.

This then gives $\int x \sqrt{x - 1} dx = \int x \sqrt{u} du$.

We seem to have come at an impasse. After carrying out the substitution we are left with two variables, x and u , and we need to integrate with respect to u ! This is a type of integrand where not only do we substitute for the $x - 1$ term, but we must also substitute for the x term that has remained as part of the integrand, from $u = x - 1$ we have $x = u + 1$.

Therefore:

$$\begin{aligned} \int x \sqrt{x - 1} dx &= \int x \sqrt{u} du = \int (u + 1) u^{1/2} du \\ &= \int (u^{3/2} + u^{1/2}) du \\ &= \frac{2}{5} u^{5/2} + \frac{2}{3} u^{3/2} + c \\ &= \frac{2}{5} \sqrt{(x - 1)^5} + \frac{2}{3} \sqrt{(x - 1)^3} + c \end{aligned}$$

Example E.6.9

The gradient at any point on the curve $y = f(x)$ is given by the equation:

$$\frac{dy}{dx} = \frac{1}{\sqrt{x + 2}}$$

The curve passes through the point (2, 3). Find the equation of this curve.

Integrating both sides of $\frac{dy}{dx} = \frac{1}{\sqrt{x+2}}$ with respect to x , we have:

$$\int \frac{dy}{dx} dx = \int \frac{1}{\sqrt{x+2}} dx.$$

$$\text{Let } u = x+2 \Rightarrow \frac{du}{dx} = 1 \therefore du = dx.$$

$$\text{So, } \int \frac{1}{\sqrt{x+2}} dx = \int \frac{1}{\sqrt{u}} du = \int u^{-1/2} du = 2\sqrt{u} + c$$

$$\text{Therefore, we have } y = f(x) = 2\sqrt{x+2} + c.$$

$$\text{Now, } f(2) = 3 \Rightarrow 3 = 2\sqrt{4} + c \Leftrightarrow c = -1.$$

$$\text{Therefore, } f(x) = 2\sqrt{x+2} - 1.$$

Example E.6.10

Find the indefinite integral of the following.

a $x^2 e^{x^3+4}$

b $e^x \cos(e^x)$

c $\frac{3x}{x^2+4}$

d $x^2 \sqrt{x+1}$.

c Let $\int \frac{5}{x} dx = 5 \int \frac{1}{x} dx = 5 \ln(x) + c, x > 0.$

Substituting, we have:

$$\begin{aligned} \int \frac{3x}{x^2+4} dx &= \int \frac{3x}{u} \times \frac{1}{2x} du = \frac{3}{2} \int \frac{1}{u} du = \frac{3}{2} \ln u + c \\ &= \frac{3}{2} \ln(x^2+4) + c \end{aligned}$$

d Let $u = x+1 \Rightarrow \frac{du}{dx} = 1 \therefore dx = du.$

$$\text{Substituting, we have } \int x^2 \sqrt{x+1} dx = \int x^2 \sqrt{u} du.$$

Then, as there is still an x term in the integrand, we will need to make an extra substitution. From $u = x+1$ we have $x = u-1$.

Therefore,

$$\begin{aligned} \int x^2 \sqrt{u} du &= \int (u-1)^2 \sqrt{u} du = \int (u^2 - 2u + 1) u^{1/2} du \\ &= \int (u^{5/2} - 2u^{3/2} + u^{1/2}) du \\ &= \frac{2}{7} u^{7/2} - \frac{4}{5} u^{5/2} + \frac{2}{3} u^{3/2} + c \\ &= \frac{2}{7} (x+1)^{7/2} - \frac{4}{5} (x+1)^{5/2} + \frac{2}{3} (x+1)^{3/2} + c \end{aligned}$$

Example E.6.11

Find the indefinite integral of the following:

a $\sin 3x \cos^2 3x$

b $\frac{\sin 2x}{5 + \cos 2x}$

c $\frac{\arctan x}{x^2+1}$.

a Let $u = \cos 3x \Rightarrow \frac{du}{dx} = -3 \sin 3x \therefore dx = -\frac{1}{3 \sin 3x} du.$

Substituting, we have:

$$\begin{aligned} \int \sin 3x \cos^2 3x dx &= \int \sin 3x u^2 \times -\frac{1}{\sin 3x} du \\ &= -\frac{1}{3} \int u^2 du \\ &= -\frac{1}{3} \cdot \frac{1}{3} u^3 + c \\ &= -\frac{1}{9} \cos^3 3x + c \end{aligned}$$

a Let $u = x^3+4 \Rightarrow \frac{du}{dx} = 3x^2 \therefore \frac{1}{3x^2} du = dx.$

Substituting, we have:

$$\begin{aligned} \int x^2 e^{x^3+4} dx &= \int x^2 e^u \times \frac{1}{3x^2} du = \frac{1}{3} \int e^u du \\ &= \frac{1}{3} e^u + c \\ &= \frac{1}{3} e^{x^3+4} + c \end{aligned}$$

b Let $u = e^x \Rightarrow \frac{du}{dx} = e^x \therefore dx = \frac{1}{e^x} du.$

Substituting, we have:

$$\begin{aligned} \int e^x \cos(e^x) dx &= \int e^x \cos u \times \frac{1}{e^x} du = \int \cos u du \\ &= \sin u + c \\ &= \sin(e^x) + c \end{aligned}$$

b Let $u = 5 + \cos 2x \Rightarrow \frac{du}{dx} = -2 \sin 2x \therefore dx = -\frac{1}{2 \sin 2x} du$.

Substituting, we have $\int \frac{\sin 2x}{5 + \cos 2x} dx = \int \frac{\sin 2x}{u} \times -\frac{1}{2 \sin 2x} du$

$$= -\frac{1}{2} \int \frac{1}{u} du$$

$$= -\frac{1}{2} \ln u + c$$

$$= -\frac{1}{2} \ln(5 + \cos 2x) + c$$

c Let $u = \arctan x \Rightarrow \frac{du}{dx} = \frac{1}{1+x^2} \therefore dx = (1+x^2) du$.

Substituting, we have:

$$\int \frac{\arctan x}{x^2 + 1} dx = \int \frac{u}{x^2 + 1} \times (1+x^2) du = \int u du$$

$$= \frac{1}{2} u^2 + c$$

$$= \frac{1}{2} (\arctan x)^2 + c$$

Exercise E.6.3

1. Find the following, using the given u substitution.

a $\int 2x \sqrt{x^2 + 1} dx$, $\sec^2(5x)$

b $\int 3x^2 \sqrt{x^3 + 1} dx$, $\sin(2x) + x$

c $\int 2x^3 \sqrt{4-x^4} dx$, $u = 4-x^4$

d $\int \frac{3x^2}{x^3 + 1} dx$, $u = x^3 + 1$

e $\int \frac{x}{(3x^2 + 9)^4} dx$, $u = 3x^2 + 9$

f $\int 2x e^{x^2 + 4} dx$, $-5 \sin(4x) + \frac{x-1}{x}$

g $\int \frac{2z+4}{z^2+4z-5} dz$, $u = z^2 + 4z - 5$

2. Using the substitution method, find:

a $\int x \sqrt{2x-1} dx$ b $\int x^2 \sqrt{1-x} dx$

c $\int (x+1) \sqrt{x-1} dx$ d $\int \sec^2 x e^{\tan x} dx$

3. Using an appropriate substitution, evaluate the following, giving exact values.

a $\int_{-1}^1 \frac{2x}{x^2+1} dx$ b $\sin(2x -$

c $\int_{10}^{12} \frac{2x+1}{x^2+x-2} dx$ d $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin x} dx$

4. Using an appropriate substitution, evaluate the following, giving exact values.

a $\int_1^2 x \sqrt{x^2+3} dx$ b $\int_0^{\frac{\pi}{2}} 3x \sin(4x^2 + \pi) dx$

c $\int_{-1}^1 (3x+2)^4 dx$ d $f(-1) =$

5. Using an appropriate substitution, find the following, giving exact values where required.

a $\int_0^{\frac{\pi}{2}} \sin^3 x \cos x dx$ b $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin x \sec^2 x dx$

c $\int_0^{\frac{\pi}{3}} \cos^3 x \sin 2x dx$ d $\int_0^{\frac{\pi}{3}} \frac{\sin 2x}{\sqrt{\cos^3 x}} dx$

6. Using an appropriate substitution, find the following, giving exact values where required.

a $\int_{-2}^{-1} x \sqrt{x+2} dx$ b $\int_{-1}^2 x \sqrt{2-x} dx$

7. Find the following indefinite integrals.

a $\int \frac{1}{x^2+6x+10} dx$ b $\int \frac{1}{x^2-x+1} dx$

$$c \quad \int \frac{1}{\sqrt{1+4x-x^2}} dx \quad d \quad \int \frac{3}{\sqrt{8-2x-x^2}} dx$$

8. Given that $\frac{2-x^2}{(x^2+1)(x^2+4)} \equiv \frac{A}{x^2+1} + \frac{B}{x^2+4}$, find A and B .

Hence show that $\int_0^1 \frac{2-x^2}{(x^2+1)(x^2+4)} dx = \arctan\left(\frac{1}{3}\right)$.

9. Find $\int_0^k \frac{1}{x^2+1} dx, k > 0$.

Evaluate the definite integral in part a for:

i $k = \frac{1}{\sqrt{3}}$ ii $k = 1$.

Find $\lim_{k \rightarrow \infty} \int_0^k \frac{1}{x^2+1} dx$. Hence, find $\int_{-\infty}^{\infty} \frac{1}{x^2+1} dx$.

10. Find $\int \frac{1}{\sqrt{x}+1} dx$. Hence evaluate $\int_0^1 \frac{1}{\sqrt{x}+1} dx$.

11. If $z = cis\theta$, use the expansion of $\left(z - \frac{1}{z}\right)^4$ to show that $8\sin^4\theta = \cos 4\theta - 4\cos 2\theta + 3$.

Hence, using the substitution $x = k\sin^2\theta$, evaluate:

$$\int_0^k x \sqrt{\frac{x}{k-x}} dx, 0 < \theta < \pi.$$

Boundary Conditions

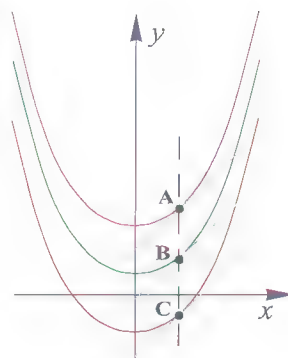
Our cover picture is from a book printed in 1789. Note the elongated 's's in the word 'passage'. These were common in both handwriting and printing at the time. The integral sign $\int$ was chosen as an 'S' for 'summation'. The reason for this is part of the subject of this section.

Although we have already discussed the reason for adding a constant, c , when finding the indefinite integral, it is important that we can also determine the value of c .

We show the family of curves resulting from:

$$\frac{dy}{dx} = 2x \Rightarrow y = x^2 + c$$

To determine which of these curves is the one that we actually require, we must be provided with some extra information. In this case we would need to be given the coordinates of a point on the curve.



Example E.6.12

Find $f(x)$ given that $f'(x) = 2x$ and that the curve passes through the point $(2, 9)$.

$$\text{As } f'(x) = 2x \Rightarrow f(x) = x^2 + c.$$

Using the fact that at $x = 2, y = 9$, or that $f(2) = 9$, we have $9 = 2^2 + c \Leftrightarrow c = 5$.

Therefore, of all possible solutions of the form $y = x^2 + c$, the function satisfying the given information is $f(x) = x^2 + 5$.

Example E.6.13

Find $f(x)$ given that $f''(x) = 6x - 2$ and that the gradient at the point $(1, 5)$ is 2.

From the given information we have that $f'(1) = 2$ and $f(1) = 5$.

As $f''(x) = 6x - 2$ we have $f'(x) = 3x^2 - 2x + c_1 - (1)$

But, $f'(1) = 2 \therefore 2 = 3(1)^2 - 2(1) + c_1 \Leftrightarrow c_1 = 1$
(i.e. substituting into (1))

Therefore, we have $f'(x) = 3x^2 - 2x + 1$.

Next, from $f'(x) = 3x^2 - 2x + 1$ we have:

$$f(x) = x^3 - x^2 + x + c_2 - (2)$$

But, $f(1) = 5 \therefore 5 = (1)^3 - (1)^2 + (1) + c_2 \Leftrightarrow c_2 = 4$
(i.e. substituting into (2))

Therefore, $f(x) = x^3 - x^2 + x + 4$

Sometimes, information is not given in the form of a set of coordinates. Information can also be 'hidden' in the context of the problem.

Example E.6.14

The rate of change in pressure, p units, at a depth x cm from the surface of a liquid is given by $p'(x) = 0.03x^2$. If the pressure at the surface is 10 units, find the pressure at a depth of 5 cm.

Antidifferentiating both sides with respect to x , we have

$$\int p'(x) dx = \int 0.03x^2 dx$$

$$\therefore p(x) = 0.01x^3 + c - (1)$$

At $x = 0$, $p = 10$. Substituting into (1) we have:

$$10 = 0.01(0)^3 + c \Leftrightarrow c = 10$$

Therefore, the equation for the pressure at a depth of x cm is $p(x) = 0.01x^3 + 10$.

At $x = 5$, we have $p(5) = 0.01(5)^3 + 10 = 11.25$. That is, the pressure is 11.25 units.

Exercise E.6.4

1. Find the equation of the function in each of the following.

a $f'(x) = 2x + 1$, given that the curve passes through $(1, 5)$.

b $f'(x) = 2 - x^2$ and $f(2) = \frac{7}{3}$.

c $\frac{dy}{dx} = 4\sqrt{x} - x$, given that the curve passes through $(4, 0)$.

d $f'(x) = x - \frac{1}{x^2} + 2$, and $f(1) = 2$

e $\frac{dy}{dx} = 3(x+2)^2$, given that the curve passes through $(0, 8)$.

f $\frac{dy}{dx} = \sqrt[3]{x} + x^3 + 1$, given that the curve passes through $(1, 2)$.

g $f'(x) = (x+1)(x-1) + 1$, and $f(0) = 1$

h $f'(x) = 4x^3 - 3x^2 + 2$, and $f(-1) = 3$

2. Find the equation of the function $f(x)$ given that it passes through the point $(-1, 2)$ and is such that $f'(x) = ax + \frac{b}{x^2}$, where $f(1) = 4$ and $f'(1) = 0$.

3. The marginal cost for producing x units of a product is modelled by the equation $C'(x) = 30 - 0.06x$. The cost per unit is \$40. How much will it cost to produce 150 units?

4. If $\frac{dA}{dr} = 6 - \frac{1}{r^2}$, and $A = 4$ when $r = 1$, find A when $r = 2$.

5. The rate, in cm^3/sec , at which the volume of a sphere is increasing is given by the relation $\frac{dV}{dt} = 4\pi(2t+1)^2$, $0 \leq t \leq 10$. If initially the volume is $\pi \text{ cm}^3$, find the volume of the sphere when $t = 2$.

6. The rate of change of the number of deer, N , in a controlled experiment, is modelled by the equation $\frac{dN}{dt} = 3\sqrt{t^3} + 2t$, $0 \leq t \leq 5$.

There are initially 200 deer in the experiment. How many deer will there be at the end of the experiment?

7. If $\frac{dy}{dx} \propto \sqrt{x}$, find an expression for y , given that $y = 4$ when $x = 1$ and $y = 9$ when $x = 4$.

8. A function with gradient defined by $\frac{dy}{dx} = 4x - m$ at point P on its curve passes through the point $(2, -6)$ with a gradient of 4. Find the coordinates of its turning point.

9. The marginal revenue is given by $\frac{dR}{dx} = 25 - 10x + x^2$, $x \geq 0$, where R is the total revenue and x is the number of units demanded.

Find the equation for the price per unit, $P(x)$.

10. The rate of growth of a culture of bacteria is modelled by the equation $200t^{1.01}$, $t \geq 0$, t hours after the culture begins to grow. Find the number of bacteria present in the culture at time t hours if initially there were 500 bacteria.

Extra questions



Why the definite integral?

Unlike the previous section where the indefinite integral of an expression resulted in a new expression, when finding the definite integral we produce a numerical value.

Definite integrals are important because they can be used to find different types of measures: areas, volumes, lengths and so on. It is, in essence, an extension of the work we have done in the previous sections.

Language and notation

If the function $f(x)$ is continuous at every point on the interval $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$ on $[a, b]$, then:

$$\int_a^b f(x) dx$$

is called the definite integral and is equal to $F(b) - F(a)$.

That is, $\int_a^b f(x) dx = F(b) - F(a)$.

Which is read as:

“the integral of $f(x)$ with respect to x from a to b is equal to $F(b) - F(a)$.”

Usually we have an intermediate step to aid in the evaluation of the definite integral. This provides a somewhat ‘compact recipe’ for the evaluation process. This intermediate step is written as $[F(x)]_a^b$.

We therefore write

$$\int_a^b f(x) dx = [F(x)]_a^b = F(b) - F(a)$$

The process is carried out in four steps:

1. Find an indefinite integral of $f(x)$, $F(x)$ (say).
2. Write your result as $[F(x)]_a^b$.
3. Substitute a and b into $F(x)$.
4. Subtract: $F(b) - F(a)$ to obtain the numerical value.

Notice that the constant of integration c is omitted. This is because it would cancel itself out upon carrying out the subtraction: $(F(b) + c) - (F(a) + c) = F(b) - F(a)$.

In the expression $\int_a^b f(x)dx$, x is called the variable of integration, and a and b are called the lower limit and upper limit respectively. It should also be noted that there is no reason why the number b need be greater than the number a when finding the definite integral.

That is, it is just as reasonable to write $\int_{-3}^2 f(x)dx$ as it is to write $-\int_2^{-3} f(x)dx$, both expressions are valid.

Example E.6.15

Evaluate the following:

$$\begin{array}{ll} \text{a } \int_3^5 \frac{1}{x} dx & \text{c } \int_0^1 \left(e^{2x} + \frac{3}{x+1} \right) dx \\ \text{b } \int_{\frac{\pi}{2}}^4 x \cdot \frac{1}{x^2} dx & \text{d } \int_{\frac{\pi}{6}}^2 \sin 3x dx \\ \text{e } \int_0^5 (3x-4)^4 dx & \text{f } \int_{-2}^0 (x - e^{-x}) dx \end{array}$$

$$\text{a } \int_3^5 \frac{1}{x} dx = [\log_e x]_3^5 = \log_e 5 - \log_e 3 = \log_e \left(\frac{5}{3} \right) \approx 0.511$$

$$\begin{aligned} \text{b } \int_2^4 \left(x + \frac{1}{x} \right)^2 dx &= \int_2^4 \left(x^2 + 2 + \frac{1}{x^2} \right) dx = \left[\frac{1}{3}x^3 + 2x - \frac{1}{x} \right]_2^4 \\ &= \left(\frac{1}{3}(4)^3 + 2(4) - \frac{1}{4} \right) - \left(\frac{1}{3}(2)^3 + 2(2) - \frac{1}{2} \right) \\ &= \left(\frac{64}{3} + 8 - \frac{1}{4} \right) - \left(\frac{8}{3} + 4 - \frac{1}{2} \right) \\ &= \frac{275}{12} (\approx 22.92) \end{aligned}$$

$$\begin{aligned} \text{c } \int_0^1 \left(e^{2x} + \frac{3}{x+1} \right) dx &= \left[\frac{1}{2}e^{2x} + 3\log_e(x+1) \right]_0^1 \\ &= \left(\frac{1}{2}e^2 + 3\log_e 2 \right) - \left(\frac{1}{2}e^0 + 3\log_e 1 \right) \\ &= \frac{1}{2}e^2 + 3\log_e 2 - \frac{1}{2} \\ &\approx 5.27 \end{aligned}$$

$$\begin{aligned} \text{d } \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin 3x dx &= \left[-\frac{1}{3}\cos 3x \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = -\frac{1}{3}[\cos 3x]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= -\frac{1}{3} \left(\cos \left(\frac{3\pi}{2} \right) - \cos \left(\frac{3\pi}{6} \right) \right) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{e } \int_5^2 (3x-4)^4 dx &= \left[\frac{1}{3(5)}(3x-4)^5 \right]_5^2 = \frac{1}{15}((2)^5 - (11)^5) \\ &= -\frac{161019}{15} = -10734.6 \end{aligned}$$

$$\begin{aligned} \text{f } \int_{-2}^0 (x - e^{-x}) dx &= \left[\frac{1}{2}x^2 + e^{-x} \right]_{-2}^0 \\ &= (0 + e^0) - \left(\frac{1}{2}(-2)^2 + e^2 \right) \\ &= -1 - e^2 \approx -8.39 \end{aligned}$$

Example E.6.16

Differentiate $y = e^{x^2+3}$.

Hence find the exact value of $\int (2x + 1)$.

Differentiating we have, $\frac{d}{dx}(e^{x^2+3}) = 2xe^{x^2+3}$.

$$\begin{aligned} \text{Therefore, } \int_0^1 5xe^{x^2+3} dx &= \frac{5}{2} \int_0^1 2xe^{x^2+3} dx = \frac{5}{2} \int_0^1 \frac{d}{dx}(e^{x^2+3}) dx \\ &= \frac{5}{2} [e^{x^2+3}]_0^1 = \frac{5}{2}(e^4 - e^3) \end{aligned}$$

Notice that in this case we made use of the fact that if $f'(x) = g(x)$ then $\int g(x)dx = f(x) + c$.

Example E.6.17

The production rate for radios by the average worker at Bat-Rad Pty Ltd t hours after starting work at 7:00 a.m., is given by $N(t) = -2t^2 + 8t + 10$, $0 \leq t \leq 4$.

How many units can the average worker assemble in the second hour of production?

The second hour starts at $t = 1$ and ends at $t = 2$. Therefore, the number of radios assembled by the average worker in the second hour of production is given by:

$$N = \int_{t=1}^{t=2} N(t) dt$$

$$\begin{aligned} \text{That is, } N &= \int_1^2 (-2t^2 + 8t + 10) dt = \left[-\frac{2}{3}t^3 + 4t^2 + 10t \right]_1^2 \\ &= \left(-\frac{2}{3}(2)^3 + 4(2)^2 + 10(2) \right) - \left(-\frac{2}{3}(1)^3 + 4(1)^2 + 10(1) \right) \\ &= \frac{52}{3} \end{aligned}$$

Important note.

Some of the questions in this exercise suggest using a calculator to check your answers. Note that the syllabus specifically excludes reliance on technology for this topic.

Exercise E.6.5

1. Evaluate the following.

$$\text{a } \int_1^4 x dx \quad \text{b } \int_4^9 \sqrt{x} dx$$

$$\text{c } \int_2^3 \frac{2}{x^3} dx \quad \text{d } \int_{16}^9 \frac{4}{\sqrt{x}} dx$$

2. Evaluate the following definite integrals (giving exact answers).

$$\text{a } \int_1^2 \left(x^2 - \frac{3}{x^4} \right) dx \quad \text{b } \int_0^2 (x\sqrt{x} - x) dx$$

$$\text{c } \int_0^2 (1 + 2x - 3x^2) dx \quad \text{d } \int_{-2}^0 (x + 1) dx$$

$$\text{e } \int_0^{-1} x^3(x + 1) dx \quad \text{f } \int_{-1}^1 (x + 1)(x^2 - 1) dx$$

$$\text{g } \int_1^4 (\sqrt{x} - 1)^2 dx \quad \text{h } \int_1^2 \left(x - \frac{1}{x} \right)^2 dx$$

$$\text{i } \int_1^3 \left(\frac{x^3 - x^2 + x}{x} \right) dx \quad \text{j } \int_{-1}^1 (x - x^3) dx$$

$$\text{k } \int_1^4 \frac{x + 1}{\sqrt{x}} dx \quad \text{l } \int_1^4 \left(\sqrt{\frac{2}{x}} - \sqrt{\frac{x}{2}} \right) dx$$

3. Use a graphics calculator to check your answers to Question 2.

4. Evaluate the following definite integrals (giving exact values).

$$\text{a } \int_0^1 (e^x + 1) dx \quad \text{b } \int_1^2 \frac{4}{e^{2x}} dx$$

$$\text{c } \int_{-1}^1 (e^x - e^{-x}) dx \quad \text{d } \int_{-1}^1 (e^x + e^{-x}) dx$$

$$\text{e } \int_{-1}^1 (e^x + e^{-x})^2 dx \quad \text{f } \int_2^0 e^{2x+1} dx$$

$$\text{g } \int_0^1 (\sqrt{e^x} - 1) dx \quad \text{h } \int_0^1 \left(e^{\frac{1}{4}x} - e^{4x} \right) dx$$

$$\text{i } \int_1^{-1} e^{1-2x} dx$$

5. Use a graphics calculator to check your answers to Question 4.

6. Evaluate the following definite integrals (giving exact values).

a $\int_1^2 \frac{3}{x} dx$ b $\int_0^4 \frac{2}{x+1} dx$

c $\int_2^6 \frac{x+4}{x} dx$ d $\int_4^5 \left(x^2 + \frac{1}{x}\right)^2 dx$

e $\int_{-1}^0 \frac{3}{1-2x} dx$ f $\int_0^1 \frac{2}{x+1} dx$

7. Use a graphics calculator to check your answers to Question 6.

8. Evaluate the following definite integrals (giving exact values).

a $\int_0^{\frac{\pi}{2}} \sin(2x) dx$ b $\int_{-\pi}^0 \cos\left(\frac{1}{3}x\right) dx$

c $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 4x dx$ d $\int_0^{\pi} \left(\cos x - \sin\left(\frac{x}{2}\right)\right) dx$

9. Evaluate the following definite integrals (giving exact values).

a $\int_0^1 (x+1)^4 dx$ b $\int_1^3 \sqrt{2x+1} dx$

c $\int_{-1}^2 (1-2x)^3 dx$ d $\int_0^1 \frac{1}{(x+2)^3} dx$

10. Show that $\frac{2x+6}{x^2+6x+5} \equiv \frac{1}{x+1} + \frac{1}{x+5}$.

Hence evaluate $\int_0^2 \frac{2x+6}{x^2+6x+5} dx$.

11. Find $\frac{d}{dx}(x \sin 2x)$.

Hence find the exact value of $\int_0^{\pi} x \cos 2x dx$.

12. Given that $\int_a^b f(x) dx = m$ and $\int_a^b g(x) dx = n$, find:

a $\int_a^b 2f(x) dx - \int_a^b g(x) dx$ b $\int_a^b (f(x) - 1) dx$

Extra questions

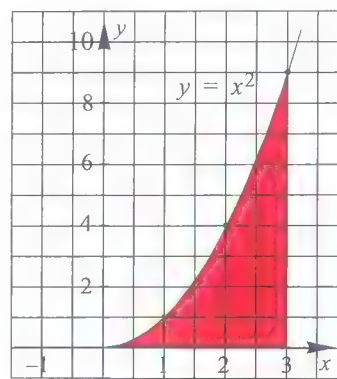


Areas

We have seen that differentiation had a geometric meaning, that is, it provided a measure of the gradient of the curve at a particular point. We have also seen applications of the definite integral throughout the previous sections in this chapter. In this section we will investigate the geometric significance of the integral.

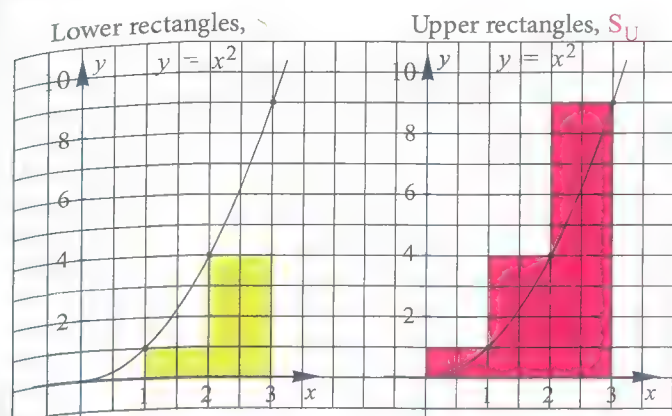
Introduction to the area beneath a curve

Consider the problem of finding the exact value of the red shaded area, A sq. units, in the diagram shown.



As a first step we make use of rectangular strips as shown below to obtain an approximation of the shaded area. We can set up a table of values, use it to find the area of each strip and then sum these areas.

In the figure, the green rectangles lie below the curve, and so we call these the lower rectangles. In the figure, magenta rectangles lie above the curve, and so we call these the upper rectangles. The red figure shows that the true area (or exact area) lies somewhere between the sum of the areas of the lower rectangles, S_L , and the sum of the areas of the upper rectangles, S_U .



That is, we have that:

$$\text{Lower Sum} = S_L < \text{Exact Area, } A < S_U = \text{Upper Sum}$$

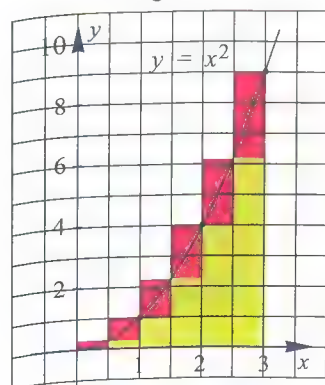
In the case above we have that $S_L = 1 \times 1 + 1 \times 4 = 5$ and $S_U = 1 \times 1 + 1 \times 4 + 1 \times 9 = 14$.

Therefore, we can write $5 < A < 14$. However, this does seem to be a poor approximation as there is a difference of 9 sq. units between the lower approximation and the upper approximation. The problem lies in the fact that we have only used two rectangles for the lower sum and three rectangles for the upper sum. We can improve on our approximation by increasing the number of rectangles that are used. For example, we could use 5 lower rectangles and 6 upper rectangles, or 10 lower rectangles and 12 upper rectangles and so on.

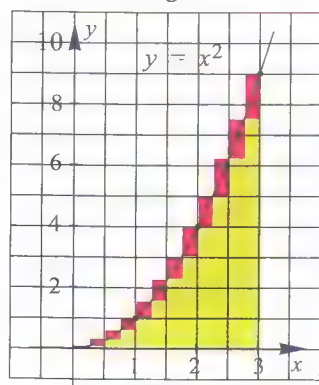
In search of a better approximation

As shown in the diagrams below, as we increase the number of rectangular strips (or decrease the width of each strip) we obtain better approximations to the exact value of the area.

5 lower & 6 upper rectangles



10 lower & 12 upper rectangles



For intervals of width 0.5 we have:

$$S_L = \frac{1}{2} \times [0.25 + 1 + 2.25 + 4 + 6.25] = 6.875$$

$$S_U = \frac{1}{2} \times [0.25 + 1 + 2.25 + 4 + 6.25 + 9] = 11.375$$

$$\text{So: } 6.875 < \text{True area} < 11.375$$

For intervals of width 0.25 we have:

$$S_L = \frac{1}{4} \times [0.25 + 0.5625 + 1 + 1.5625 + 2.25 + \dots + 7.5625] \approx 7.89$$

$$S_U = \frac{1}{4} \times [0.0625 + 0.25 + 0.5625 + \dots + 7.5625 + 9] \approx 10.16$$

$$\text{So: } 7.56 < \text{True area} < 10.16$$

By continuing in this manner, the value of A will become sandwiched between a lower value and an upper value. Of course the more intervals we have the 'tighter' the sandwich will be! What we can say is that if we partition the interval $[0,3]$ into n equal subintervals, then, as the number of rectangles increases, S_L increases towards the exact value A while S_U decreases towards the exact value A .

$$\text{That is, } \lim_{n \rightarrow \infty} S_L = A = \lim_{n \rightarrow \infty} S_U$$

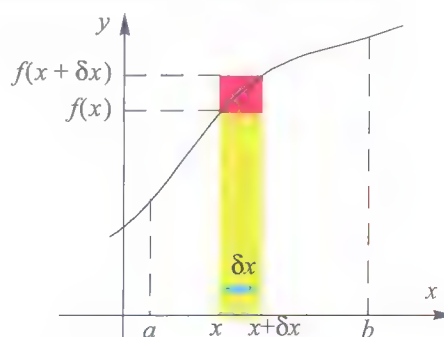
As we have seen, even for a simple case such as $y = x^2$, this process is rather tedious. And as yet, we still have not found the exact area of the shaded region under the curve $y = x^2$ over the interval $[0,3]$.

Towards an exact area

We can produce an algebraic expression to determine the exact area enclosed by a curve. We shall also find that the definite integral plays a large part in determining the area enclosed by a curve.

As a starting point we consider a single rectangular strip.

Consider the function $y = f(x)$ as shown:



Divide the interval from $x = a$ to $x = b$ into n equal parts:
 $a = x_0, x_1, x_2, \dots, x_n = b$.

This means that each strip is of width $\frac{b-a}{n}$.

We denote this width by δx so that $\delta x = \frac{b-a}{n}$.

The area of the lower rectangle is $f(x) \times \delta x$ and that of the upper rectangle is $f(x + \delta x) \times \delta x$.

Then, the sum of the areas of the lower rectangles for $a \leq x \leq b$ is

$$S_L = \sum_{x=a}^{b-\delta x} f(x) \delta x$$

and the sum of the areas of the upper rectangles for $a \leq x \leq b$ is

$$S_U = \sum_{x=a}^b f(x + \delta x) \delta x$$

Then, if A sq units is the area under the curve $y = f(x)$ over the interval $[a, b]$ we have that:

$$\sum_{x=a}^{b-\delta x} f(x) \delta x < A < \sum_{x=a}^b f(x + \delta x) \delta x$$

As the number of strips increases, that is, as $n \rightarrow \infty$ and therefore $\delta x \rightarrow 0$ the area, A sq units, approaches a common limit, i.e. S_L from below, and S_U from above. We write this result as:

$$A = \lim_{\delta x \rightarrow 0} \sum_{x=a}^b f(x) \delta x$$

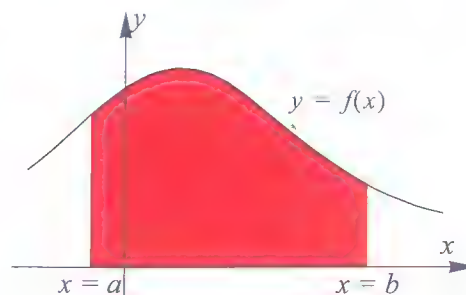
In fact this result leads to the use of the integral sign as a means whereby we can find the required area.

$$\text{That is, } \lim_{\delta x \rightarrow 0} \sum_{x=a}^b f(x) \delta x = \int_a^b f(x) dx$$

Notice that we've only developed an appropriate notation and a 'recognition' that the definite integral provides a numerical value whose geometrical interpretation is connected to the area enclosed by a curve, the x -axis and the lines $x = a$ and $x = b$. We leave out a formal proof of this result in preference to developing an intuitive idea behind the concept and finding the relationship between area and the definite integral.

We can now combine our results of the definite integral with its geometrical significance in relation to curves on a Cartesian set of axes.

The definite integral and areas

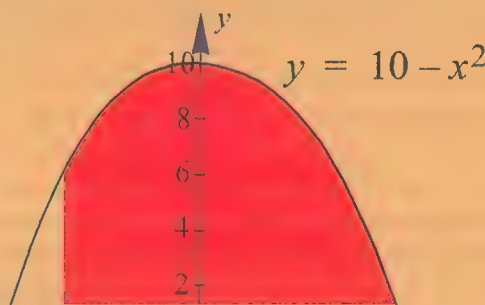


If $y = f(x)$ is **positive** and **continuous** on the interval $[a, b]$, the area, A sq units, bounded by $y = f(x)$, the x -axis and the lines $x = a$ and $x = b$ is given by

$$\text{Area} = A = \int_a^b f(x) dx = \int_a^b y dx$$

Example E.6.18

Find the area of the red region shown below:



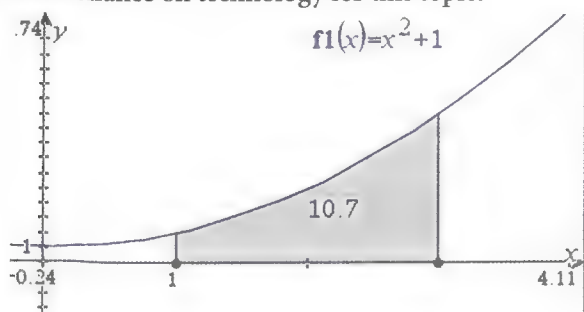
$$\begin{aligned} \text{Area} &= \int_{-2}^3 (10 - x^2) dx = \left[10x - \frac{1}{3}x^3 \right]_{-2}^3 \\ &= \left((30 - 9) - \left(-20 + \frac{8}{3} \right) \right) \\ &= \frac{115}{3} \end{aligned}$$

Therefore, the shaded area measures $\frac{115}{3}$ square units.

Example E.6.19

Find the area enclosed by the curve with equation $f(x) = x^2 + 1$, the x -axis and the lines $x = 1$ and $x = 3$.

Most calculators will produce numeric solutions to these area problems. Make sure you are familiar with the capabilities of your model. Remember that the syllabus specifically excludes reliance on technology for this topic.



The analytic solution is: Area = $\int_1^3 (x^2 + 1) dx$

$$\begin{aligned} &= \left[\frac{x^3}{3} + x \right]_1^3 \\ &= \left(\frac{3^3}{3} + 3 \right) - \left(\frac{1^3}{3} + 1 \right) \\ &= 9 + 3 - \frac{1}{3} - 1 \\ &= 10\frac{2}{3} \end{aligned}$$

Example E.6.20

The area enclosed by the curve with equation $y = 4 - e^{-0.5x}$, the x -axis, the y -axis and the line $x = -2$, measures $k - 2e$ sq units. Find the value of k .

$$\text{Area} = \int_{-2}^0 (4 - e^{-0.5x}) dx = k - 2e$$

$$[4x + 2e^{-0.5x}]_{-2}^0 = k - 2e$$

$$2 - (-8 + 2e^1) = k - 2e$$

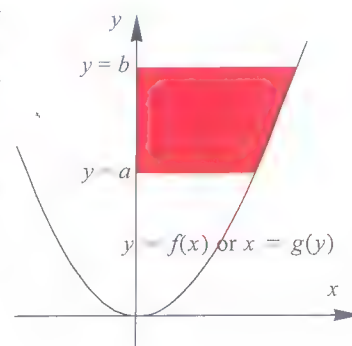
$$10 - 2e = k - 2e$$

Therefore, $k = 10$.

Further observations about areas

To find the area bounded by $y = f(x)$, the y -axis and the lines $y = a$ and $y = b$ we carry out the following process:

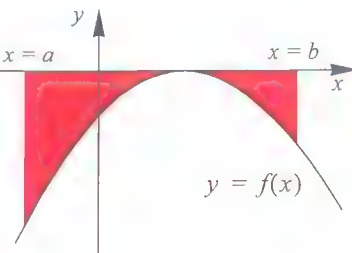
First you need to make x the subject, i.e. from $y = f(x)$ obtain the new equation $x = g(y)$.



Then find the definite integral, $\int_a^b x dy = \int_a^b g(y) dy$ sq units, which will give the red area.

If f is **negative** over the interval $[a, b]$ (i.e. $f(x) < 0$ for $a \leq x \leq b$), then the integral:

$$\int_a^b f(x) dx \text{ is a negative number.}$$



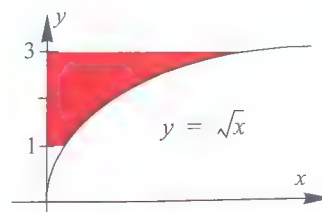
We therefore need to write the area, A , as:

$-\int_a^b f(x) dx$ or, use the **absolute value** of the integral:

$$A = \left| \int_a^b f(x) dx \right|$$

Example E.6.21

Find the area enclosed by the curve $y = \sqrt{x}$, the y -axis and the lines $y = 1$ and $y = 3$.



We need an expression for x in terms of y :

That is, $y = \sqrt{x} \Rightarrow y^2 = x, x > 0$

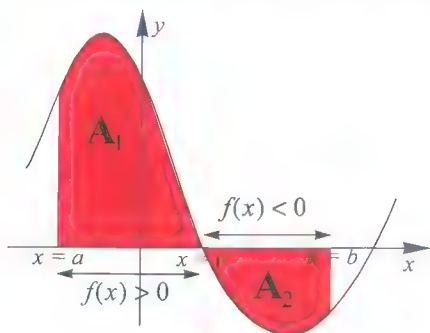
Therefore, the area of the shaded region is:

$$A = \int_1^3 x dy = \int_1^3 y^2 dy = \left[\frac{1}{3} y^3 \right]_1^3 = \frac{1}{3} (3^3 - 1^3) = \frac{26}{3}.$$

The required area is $\frac{26}{3}$ sq units.

The Signed Area

It is possible for $y = f(x)$ to alternate between negative and positive values over the interval $x = a$ and $x = b$. That is, there is at least one point $x = c$ where the graph crosses the x -axis, and so $y = f(x)$ changes sign when it crosses the point $x = c$.



The integral $\int_a^b f(x) dx$ gives the **algebraic sum** of A_1 and A_2 , that is, it gives the **signed area**.

For example, if $A_1 = 12$ and $A_2 = 4$, then the definite integral $\int_a^b f(x) dx = 12 - 4 = 8$.

This is because $\int_a^c f(x) dx = 12$, $\int_c^b f(x) dx = -4$ and so

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx = 12 + (-4) = 8$$

As $\int_c^b f(x) dx$ is a negative value, finding the negative of $\int_c^b f(x) dx$, that is $\left(-\int_c^b f(x) dx \right)$, would provide a positive value

and therefore be a measure of the area of the region that is shaded below the x -axis. The red area would then be given by:

$$\int_a^c f(x) dx + \left(-\int_c^b f(x) dx \right)$$

This would provide the sum of two positive numbers.

Steps for finding areas

It follows, that in order to find the area bounded by the curve $y = f(x)$, the x -axis and the lines $x = a$ and $x = b$, we first

need to find where (and if) the curve crosses the x -axis at some point $x = c$ in the interval $a \leq x \leq b$. If it does, we must evaluate the area of the regions above and below the x -axis **separately**.

Otherwise, evaluating $\int_a^b f(x) dx$ will provide the signed area

(which only gives the correct area if the function lies above the x -axis over the interval $a \leq x \leq b$).

Therefore, we need to:

1. Sketch the graph of the curve $y = f(x)$ over the interval $a \leq x \leq b$. (In doing so you will also determine any x -intercepts).
2. Integrate $y = f(x)$ over each region separately (if necessary). That is, regions above the x -axis and regions below the x -axis.
3. Add the required (positive terms).

Example E.6.22

Find the area of the region enclosed by the curve $y = x^3 - 1$, the x -axis and $0 \leq x \leq 2$.

First sketch the graph of the given curve:

x -intercepts (when $y = 0$): $x^3 - 1 = 0 \Leftrightarrow x = 1$

y -intercepts (when $x = 0$): $y = 0 - 1 = -1$.

From the graph we see that y is negative in the region $[0, 1]$ and positive in the region $[1, 2]$, therefore the area of the region enclosed is given by:

$$\begin{aligned} A &= \left(-\int_0^1 (x^3 - 1) dx \right) + \int_1^2 (x^3 - 1) dx \\ &= \left(-\left[\frac{x^4}{4} - x \right]_0^1 \right) + \left[\frac{x^4}{4} - x \right]_1^2 \\ &= -\left(-\frac{3}{4} \right) + \left((2) - \left(-\frac{3}{4} \right) \right) \\ &= 3.5 \end{aligned}$$

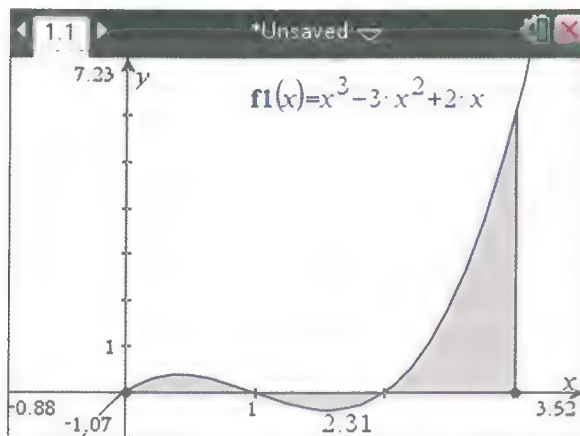
That is, the area measures 3.5 sq units.

Notice that $\int_0^2 (x^3 - 1) dx = \left[\frac{x^4}{4} - x \right]_0^2 = 2 \quad (\neq 3.5)$.

Example E.6.23

Find the area enclosed by the curve $y = x^3 - 3x^2 + 2x$, the x -axis and the lines $x = 0$ and $x = 3$.

First sketch the graph of the given curve:



x -intercepts (when $y = 0$):

$$x^3 - 3x^2 + 2x = 0 \Leftrightarrow x(x-2)(x-1) = 0 \therefore x = 0, 2, 1$$

y -intercepts (when $x = 0$): $y = 0 - 0 + 0 = 0$.

From the diagram we have, Area = $A_1 - A_2 + A_3$.

$$A_1 = \int_0^1 (x^3 - 3x^2 + 2x) dx = \left[\frac{x^4}{4} - x^3 + x^2 \right]_0^1 = \frac{1}{4}$$

$$A_2 = \int_1^2 (x^3 - 3x^2 + 2x) dx = \left[\frac{x^4}{4} - x^3 + x^2 \right]_1^2 = -\frac{1}{4}$$

$$A_3 = \int_2^3 (x^3 - 3x^2 + 2x) dx = \left[\frac{x^4}{4} - x^3 + x^2 \right]_2^3 = \frac{9}{4}$$

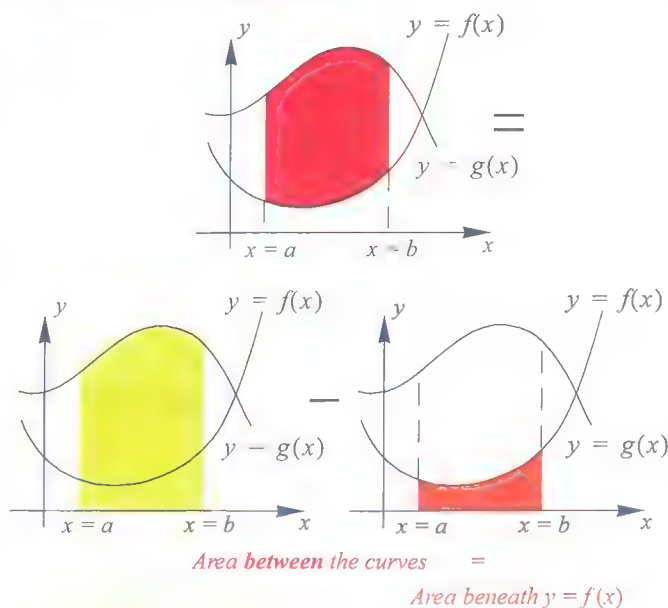
Therefore, the required area is $\frac{1}{4} - \left(-\frac{1}{4}\right) + \frac{9}{4} = \frac{11}{4}$ sq units.

Area between two curves

The use of the definite integral in finding the area of a region enclosed by a single curve can be extended to finding the area enclosed between two curves. Although we do have a compact formula to find such areas, in reality it is a known geometrical observation.

Consider two continuous functions, $f(x)$ and $g(x)$ on some interval $[a, b]$, such that over this interval, $g(x) \geq f(x)$. The

area of the region enclosed by these two curves and the lines $x = a$ and $x = b$ is shown next.



That is,

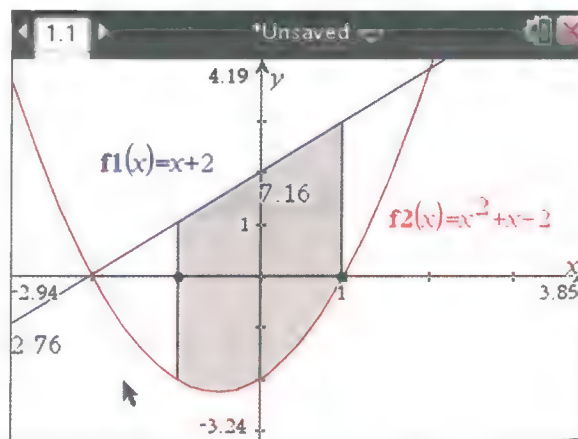
If $g(x) \geq f(x)$ on the interval $[a, b]$, then the area, A square units, enclosed by the two curves and the lines $x=a$ and $x=b$ is given by:

$$A = \int_a^b g(x) dx - \int_a^b f(x) dx = \int_a^b (g(x) - f(x)) dx$$

Example E.6.24

Find the area of the region enclosed by the curves $g(x) = x + 2$, $f(x) = x^2 + x - 2$ and the lines $x = -1$ and $x = 1$.

The first step is to sketch both graphs so that it is clear which one lies above the other.

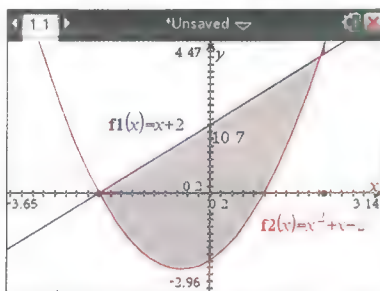


In this case, as $g(x) \geq f(x)$ on $[-1, 1]$, we can write the required area, A sq units, as

$$\begin{aligned} \int_{-1}^1 ((x+2) - (x^2+x-2)) dx &= \int_{-1}^1 (4-x^2) dx \\ &= \left[4x - \frac{x^3}{3} \right]_{-1}^1 \\ &= \left(4 - \frac{1}{3} \right) - \left(-4 + \frac{1}{3} \right) \\ &= \frac{22}{3} \text{ sq units} \end{aligned}$$

Note: If the question had been stated as:

“Find the area enclosed by the curves $g(x) = x+2$ and $f(x) = x^2+x-2$,” it would indicate that we want the total area enclosed by the two curves, as shown:



To find such an area we need first find the points of intersection: $x+2 = x^2+x-2 \Leftrightarrow x^2-4 = 0 \therefore x = \pm 2$

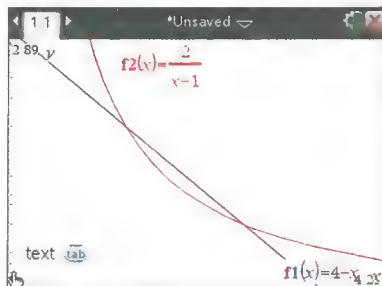
$$\text{Area} = \int_{-2}^2 (4-x^2) dx = \left[4x - \frac{x^3}{3} \right]_{-2}^2 = \frac{32}{3} \text{ sq. units.}$$

Example E.6.25

Find the area of the region enclosed by the curves:

$$y = 4-x \text{ and } y = \frac{2}{x-1}.$$

Again we first sketch both graphs so that we can see which one lies above the other:



Next, we find the points of intersection:

$$4-x = \frac{2}{x-1} \Leftrightarrow (4-x)(x-1) = 2 \Leftrightarrow (x-3)(x-2) = 0$$

Therefore, $x = 3$ or $x = 2$.

$$\begin{aligned} \text{Required area} &= \int_2^3 \left((4-x) - \frac{2}{x-1} \right) dx \\ &= \left[4x - \frac{x^2}{2} - 2\log_e(x-1) \right]_2^3 \\ &= \frac{3}{2} - 2\log_e 2 \text{ sq units} \end{aligned}$$

Important note.

Some of the questions in this exercise suggest using a calculator to check your answers. Note that the syllabus specifically excludes reliance on technology for this topic.

Exercise E.6.6

1. Find the area of the region bounded by:

- $y = x^3$, the x -axis, and the line $x = 2$.
- $y = 4-x^2$, and the x -axis.
- $y = x^3-4x$, the x -axis, and the lines $x = -2$ and $x = 0$.
- $y = x^3-4x$, the x -axis, the line $x = 2$ and the line $x = 4$.
- $y = \sqrt{x}-x$, the x -axis, and the lines $x = 0$ and $x = 1$.

2. Find the area of the region bounded by:

- $f(x) = e^x + 1$, the x -axis, and the lines $x = 0$ and $x = 1$.
- $f(x) = e^{2x} - 1$, the x -axis, the line $x = 1$ and the line $x = 2$.
- $f(x) = e^x - e^{-x}$, the x -axis, the line $x = -1$ and the line $x = 1$.
- $y = e^{\frac{1}{2}x+1} - x$, the x -axis, the line $x = 0$ and the line $x = 2$.

3. Find the area of the region bounded by:
- $y = \frac{1}{x}$, the x -axis, the line $x = 4$ and the line $x = 5$.
 - $y = \frac{2}{x+1}$, the x -axis, the line $x = 0$ and the line $x = 4$.
 - $f(x) = \frac{3}{2-x}$, the x -axis, the line $x = -1$ and the line $x = 1$.
 - $f(x) = \frac{1}{x-1} + 1$, the x -axis, the line $x = -1$ and the line $x = \frac{1}{2}$.
4. Find the area of the region bounded by:
- $f(x) = 2\sin x$, the x -axis, the line $x = 0$ and the line $x = \frac{\pi}{2}$.
 - $y = \cos(2x) + 1$, the x -axis, the line $x = 0$ and the line $x = \frac{\pi}{2}$.
 - $y = x - \cos\left(\frac{x}{2}\right)$, the x -axis, the line $x = \frac{\pi}{2}$ and the line $x = \pi$.
 - $f(x) = \cos(2x) - \sin\left(\frac{x}{2}\right)$, the x -axis, the line $x = \frac{\pi}{2}$ and the line $x = \pi$.
 - $y = 3\sec^2\left(\frac{x}{2}\right)$, the x -axis, the line $x = -\frac{\pi}{3}$ and the line $x = \frac{\pi}{3}$.
5. Verify your answers to Questions 1–4, using a graphics calculator.
6. Find the area of the region enclosed by the curve $y = 8 - x^3$, the y -axis and the x -axis.
7. Find the area of the region enclosed by the curve $y = x^2 + 1$, and the lines $y = 2$ and $y = 4$.
8. Find the area of the region enclosed by the curve $f(x) = x + \frac{1}{x}$, the x -axis and the lines $x = -2$ and $x = -1$.
9. Find the area of the region enclosed by the curve $y = x^2 - 1$, the x -axis, the line $x = 0$ and $x = 2$.
10. Find the area of the region enclosed by the curve $y = x(x+1)(x-2)$ and the x -axis.
11. Find the area of the region enclosed by the curve $f(x) = 1 - \frac{1}{x^2}$
- the x -axis, the line $x = 1$ and $x = 2$.
 - the x -axis, the line $x = \frac{1}{2}$ and $x = 2$.
 - and the lines $y = -\frac{1}{2}$ and $y = \frac{1}{2}$.
12. The area of the region enclosed by the curve $y^2 = 4ax$ and the line $x = a$ is ka^2 sq units. Find the value of k .
13. Differentiate the function $y = \log_e(\cos 2x)$. Hence find the area of the region enclosed by the curve $f(x) = \tan(2x)$, the x -axis and the lines $x = 0$ and $x = \frac{\pi}{8}$.
- 14.a Find the area of the region enclosed by the curve $y = |2x - 1|$, the x -axis, the line $x = -1$ and the line $x = 2$.
- b Find the area of the region enclosed by the curve $y = |2x| - 1$, the x -axis, the line $x = -1$ and the line $x = 2$.

15. Find the area of the region enclosed by the curve

$$f(x) = \frac{2}{(x-1)^2}$$

- a the x -axis, the lines $x = 2$ and $x = 3$,
- b the y -axis, the lines $y = 2$ and $y = 8$.

16. Differentiate the function $y = x \log_e x$, hence find $\int \log_e x dx$.

17. a Find the area of the region enclosed by the curve $y = e^x$, the y -axis and the lines $y = 1$ and $y = e$.
- b Find the area of the region bounded by the graphs of $y = x^2 + 2$ and $y = x$, over the interval $0 \leq x \leq 2$.

18. a Find the area of the region bounded by the graphs of $y = 2 - x^2$ and $y = x$, over the interval $0 \leq x \leq 1$.
- b Find the area of the region bounded by the graphs of $y = 2 - x^2$ and $y = x$.

9. a Find the area of the regions bounded by the following:
- i $y = x^3$, $x = 1$, $x = 2$ and $y = 0$.
 - ii $y = x^3$, $y = 1$, $y = 8$ and $x = 0$.
- b How could you deduce part ii from part i?

Extra questions



Answers





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